Доклади на Българската академия на науките
Comptes rendus de l'Académie bulgare des Sciences

Tome 66, No 3, 2013

MATHEMATIQUES
Algèbre

SERIES OF VARIETIES OF LIE ALGEBRAS OF DIFFERENT FRACTIONAL EXPONENTS

Olga Malyusheva, Sergey Mishchenko, Andrey Verevkin

(*Submitted by Corresponding Member V. Drensky on September 24, 2012*)

Abstract

We consider numerical characteristics of varieties of Lie algebras over a field of characteristic zero and their polynomial identities. Here we have constructed an infinite series of such varieties with different fractional exponents. This extends the special cases known before.

Key words: variety of Lie algebras, polynomial identity, exponent of variety

2010 Mathematics Subject Classification: 17B01, 16R10, 16P90

Let $\mathbf{V}$ be a variety of linear algebras over a field of zero characteristic and let $F(\mathbf{V})$ be its free algebra on a countable set $X = \{x_1, x_2, \ldots\}$. If $P_n(\mathbf{V})$ is the vector subspace of $F(\mathbf{V})$ consisting of the multilinear polynomials in the first n variables, then the n-th codimension $c_n(\mathbf{V})$ of $\mathbf{V}$ is the dimension of $P_n(\mathbf{V})$, i.e., the dimension of the multilinear polynomials of degree n in the absolutely free non-associative algebra modulo the polynomial identities satisfied by $\mathbf{V}$. The growth of the codimension sequence $c_n(\mathbf{V})$, $n = 1, 2, \ldots$, is called the growth of the variety $\mathbf{V}$. If $c_n(\mathbf{V})$ is majorized by the exponent a^n for an appropriate a, then there exist limits

$$\mathrm{LEXP}(\mathbf{V}) = \liminf_{n\to\infty} \sqrt[n]{c_n(\mathbf{V})}, \quad \mathrm{HEXP}(\mathbf{V}) = \limsup_{n\to\infty} \sqrt[n]{c_n(\mathbf{V})}.$$

We shall call them the lower and upper exponents of the variety $\mathbf{V}$, respectively. If the limit of the sequence $\sqrt[n]{c_n(\mathbf{V})}$ exists, then we call it the exponent of $\mathbf{V}$,

$$\mathrm{EXP}(\mathbf{V}) = \mathrm{LEXP}(\mathbf{V}) = \mathrm{HEXP}(\mathbf{V}).$$

The symmetric group S_n acts on the space $P_n(\mathbf{V})$ by permuting the indices of the variables. If $x_{i_1} \cdots x_{i_n} \in P_n(\mathbf{V})$ and $p \in S_n$, then $p(x_{i_1} \cdots x_{i_n}) = x_{p(i_1)} \cdots x_{p(i_n)}$, and $P_n(\mathbf{V})$ becomes an S_n-module. The character of the S_n-module $P_n(\mathbf{V})$ is decomposed into a linear combination of irreducible characters

$$\chi(P_n(\mathbf{V})) = \sum_{\lambda \vdash n} m_\lambda \chi_\lambda, \tag{1}$$

where χ_λ denotes the character of the irreducible S_n-module corresponding to the partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ of n and the nonnegative integer m_λ is the multiplicity of χ_λ. Then $c_n(\mathbf{V}) = \sum_{\lambda \vdash n} m_\lambda d_\lambda$, where $d_\lambda = \deg \chi_\lambda$ is the degree of the character χ_λ.

Any variety of associative algebras has at most exponential growth and its exponent has a natural value (see [1] and [2]). In the general case, it was shown in [3] that for any real number $\alpha > 1$ there exists a variety $\mathbf{V}_\alpha$ such that $\mathrm{EXP}(\mathbf{V}_\alpha) = \alpha$. The first example of a variety of Lie algebras with fractional exponent was constructed in [4] and the approximate value of its exponent was calculated in [5]. Another example of a variety with fractional exponent was constructed in [6]. In the sequel we shall follow the work [7], where the necessary background was stated.

We shall use left-normed arranging in the Lie products and shall omit the brackets, e.g., $(ab)c \equiv abc$. The bar or the tilde are used to denote the alternation of the generators. Capital letters denote the inner derivation, i.e., $\mathrm{ad}\, y(x) = xY = xy$. For example,

$$y_1 \overline{X}_1 [\overline{X}_2, \overline{Y}] = 2(y_1 x_1 x_2 y + y_1 x_2 y x_1 + y_1 y x_1 x_2 - y_1 x_1 y x_2 - y_1 y x_2 x_1 - y_1 x_2 x_1 y),$$
$$\overline{X}_1 [\overline{X}_2, \overline{X}_3][[\overline{X}_4, \overline{X}_5], Y] = \sum_{p \in S_5} (-1)^p X_{p(1)} [X_{p(2)}, X_{p(3)}][[X_{p(4)}, X_{p(5)}], Y],$$

where S_n is the symmetric group and $(-1)^p$ is the parity of the permutation $p \in S_n$.

Let $\mathbf{A}^2$ be the variety of all metabelian Lie algebras determined by the identity

$$(x_1 x_2)(x_3 x_4) \equiv 0,$$

and let $M_s = F_s(\mathbf{A}^2)$ denote the relatively free algebra of this variety over the set of free generators $\{z_1, z_2, \ldots, z_s\}$. Consider the linear transformation d_s of the vector space spanned by $z_1, z_2, \ldots, z_s$ defined by the rule $z_i d_s = z_{i+1}$, $i = 1, 2, \ldots, s-1$, $z_s d_s = 0$. Then d_s can be extended to a derivation of the algebra M_s, denoted by the same letter. Let $\langle d_s \rangle$ be the one-dimensional Lie algebra generated by d_s (with trivial multiplication). We may built the semidirect product $L_s = M_s \leftthreetimes \langle d_s \rangle$. The variety generated by the algebra L_s is denoted by $\mathrm{var}(L_s)$, $s \in \mathbb{N}$.

Theorem. *For the varieties of Lie algebras* $\mathrm{var}(L_s)$, $s \in \mathbb{N}$, *over a field of zero characteristic the following strict inequalities hold*

$$3 < \exp(L_3) < \exp(L_4) < \cdots < \exp(L_s) < \exp(L_{s+1}) < \cdots < 4.$$

The proof of the theorem will require the following statement.

Lemma 1. *In the sum* (1) *the multiplicity* m_λ *vanishes if the Young diagram of the partition* $\lambda = (\lambda_1, \ldots, \lambda_k)$ *of* n *has at least two cells below the first* $s+1$ *rows. Also for* $m_\lambda \neq 0$ *the lengths of the rows satisfy the following inequality*:

$$\sum_{i=1}^{s+1} (2-i) \cdot \lambda_i + s(s-1) \geq 0.$$

Proof. Assume the contrary. Let us suppose that $\lambda \vdash n$ has more than two cells below the first $s+1$ rows in the corresponding Young diagram. Consider an element f that generates an irreducible module corresponding to λ. Let $\lambda'_1, \ldots, \lambda'_{l(\lambda)}$ be the heights of the columns of this diagram.

According to [8], the element f is equal to a linear combination of components, where every component contains $l(\lambda)$ skew-symmetric sets with λ'_i variables in the i-th set. So, it is sufficient to prove that any multilinear Lie polynomial, containing either $s+3$ skew-symmetric variables or two sets with $s+2$ skew-symmetric variables, identically equals zero. This is clear because the algebra L_s contains the abelian ideal M_s^2 of codimension $s+1$. Hence the lemma is true for diagrams with two or more cells below the first $s+1$ rows.

Let us prove the second statement about the restrictions on the lengths of the rows of the diagram. Consider $\lambda = (\lambda_1, \lambda_2, \ldots) \vdash n$ such that $\sum_{i=1}^{s+1} \lambda_i \geq n-1$ and $\sum_{i=1}^{s+1} (2-i) \cdot \lambda_i + s(s-1) < 0$. We shall show that this partition determines a polynomial identity of the algebra L_s. It is sufficient to prove that any multilinear polynomial f depending upon $l = \lambda_1$ skew-symmetric sets of variables with $\lambda'_1, \ldots, \lambda'_l$ elements, respectively, takes only zero value in L_s.

Let $d, z_1, z_2, \ldots$ be a basis of the algebra L_s. We shall replace the variables of f with some of these elements. The element d can participate not more than once in each skew-symmetric set of variables, otherwise f will be equal to zero. Let us identify the variables in f that we substitute by d, and denote them by b. The other variables will be denoted by $y_1, \ldots, y_k$. Taking into account that d is a derivation of the algebra M_s, we can rewrite the polynomial f as a linear combination of the following products:

$$(2) \qquad (y_{s_1} b^{\alpha_1})(y_{s_2} b^{\alpha_2}) \cdots (y_{s_k} b^{\alpha_k}),$$

in which $\alpha_1, \ldots, \alpha_k \geq 0$. Note also that $\alpha_1 + \cdots + \alpha_k \leq \lambda_1$.

The elements $y_i b^{\alpha_i}$ may be considered as new variables. Although the polynomial f is not multilinear in these variables, it may be written as a sum $f = f_1 + \cdots + f_m$, such that each f_i is a multilinear polynomial in some of the new variables. Now we shall prove that each component $f_1, \ldots, f_m$ takes zero value. For this purpose we write for example f_1 as a linear combination of elements of the form (2), then fix the indices s_1, s_2, and show that the partial sum f_1^1 of f_1 for these fixed s_1, s_2 at the first two positions equals zero.

If f is skew-symmetric in y_1 and y_2, then the elements from $f_1, \ldots, f_m$, which depend on $y_1 b$, $y_2 b$ are also skew-symmetric in these variables. Similarly, the skew-symmetry in $y_1 b^j$, $y_2 b^j$, where $j = 2, \ldots, s-1$ will be preserved. Let f be skew-symmetric in the variables $y_1, y_2, \ldots, y_s$ which are different from y_{s_1}, y_{s_2} and let y_1, y_2 participate in f_1^1 as $y_1 b^j$, $y_2 b^j$ with the same $j = 1, \ldots s-1$. If p is a permutation of $s_3, \ldots, s_k$, then

$$y_{s_1} y_{s_2} y_{p(s_3)} \cdots y_{p(s_k)} = y_{s_1} y_{s_2} y_{s_3} \cdots y_{s_k}$$

in the free metabelian algebra M_s. Since f_1^1 is skew-symmetric in $y_1 b^j$, $y_2 b^j$, we derive that f_1^1 will have zero values only. In other words, if f_1^1 depends on $y_1 b^{\alpha_1}, y_2 b^{\alpha_2}, \ldots, y_i b^{\alpha_i}$ and takes a non-zero value, then all $\alpha_1, \alpha_2, \ldots, \alpha_i$ are pairwise different and $\alpha_1 + \alpha_2 + \cdots + \alpha_i \geq i(i-1)/2$, $i = 2, \ldots, s$. Recall that the original polynomial f depends on λ_{s+1} skew-symmetric sets of cardinality $\leq s+1$, and depends on $\lambda_i - \lambda_{i+1}$ skew-symmetric sets of cardinality i, where $i = 3, \ldots, s$. The variables y_{s_1} and y_{s_2} are used not more than twice in these sets and it is possible to substitute by d only one variable from each set. So we have at most $\lambda_{s+1} - 2$ skew-symmetric sets with s elements and $\lambda_i - \lambda_{i+1}$ sets with $i-1$ elements, where $i = 3, \ldots, s$. We have shown above that f_1^1 may take non-zero values only if the following condition holds:

$$\alpha_1 + \cdots + \alpha_k \geq \sum_{i=3}^{s} \frac{(i-1)(i-2)}{2} \cdot (\lambda_i - \lambda_{i+1}) + \frac{s(s-1)}{2}(\lambda_{s+1} - 2).$$

But $\alpha_1 + \cdots + \alpha_k \leq \lambda_1$ implies the inequality $\sum_{i=1}^{s+1}(2-i) \cdot \lambda_i + s(s-1) \geq 0$. It means that if $\sum_{i=1}^{s+1}(2-i) \cdot \lambda_i + s(s-1) < 0$ then f_1^1 takes zero values only and the same holds for f_1 and f. Our lemma is proved. $\square$

Proof of the Theorem. Let $\mathbf{V} = \mathrm{var}(L_s)$. Lemma 1 implies that if a diagram has two and more cells below the first $s+1$ rows or if $\sum_{i=1}^{s+1}(2-i) \cdot \lambda_i + s(s-1) < 0$, then the multiplicity m_λ is equal to 0. In particular, the variety satisfies the system of Capelli identities. As it is proved in [9], in this case, the multiplicity of m_λ is polynomially bounded. It is clear that the system of Capelli identities implies that the number of terms in (1) is also polynomially bounded. Therefore, the upper and lower limit of the exponential functions can be found by analyzing the dimensions of the irreducible modules of the symmetric group.

Since a limited number of cells does not affect the numerical values of the upper and lower exponents of the variety, we consider a partition into no more than $s+1$ parts.

Let $\alpha_i = \lambda_i/n$, $i = 1, \dots, s+1$, be numbers corresponding to the partition $\lambda = (\lambda_1, \dots, \lambda_{s+1}) \vdash n$, where some of the last λ_i may equals zero. We consider the function

$$F(\alpha_1, \alpha_2, \dots, \alpha_{s+1}) = \prod_{i=1}^{s+1} \alpha_i^{-\alpha_i},$$

where $0^0 = 1$. For any positive integer t we consider the partition $\lambda(t) = (\alpha_1 nt, \alpha_2 nt, \dots, \alpha_{s+1} nt)$. Let $d_{\lambda(t)}$ be the dimension of the module of the symmetric group S_{nt} corresponding to the partition $\lambda(t) \vdash nt$. By the hook-formula for the dimension of the irreducible representations of the symmetric group and the Stirling formula for the factorials, we obtain

$$\lim_{t\to\infty} \sqrt[nt]{d_{\lambda(t)}} = F(\alpha_1, \alpha_2, \dots, \alpha_{s+1}).$$

Let T be the domain of $\mathbb{R}^{s+1}$ defined by the conditions

$$\left\{ \begin{array}{c} \alpha_1 + \alpha_2 + \cdots + \alpha_{s+1} = 1 \\ \displaystyle\sum_{i=1}^{s+1} (2-i)\cdot \alpha_i \geq 0 \\ \alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_{s+1} \geq 0. \end{array} \right. \tag{3}$$

In the second condition, we ignore the term $s(s-1)$, since it does not affect the maximum value of the function.

Since the function $F(\overrightarrow{\alpha})$ is continuous, it attains its maximal value on the compact set T at some point $\overrightarrow{\alpha}^{(0)} \in T$: $F_{\max} = F(\overrightarrow{\alpha}^{(0)}) = \max_{\overrightarrow{\alpha}\in T} F(\overrightarrow{\alpha})$.

This together with the fact that the number of terms and the multiplicities in (1) are polynomially bounded, we conclude that $\mathrm{HEXP}(\mathbf{V}) \leq F_{\max}$.

For the proof of the inequality $\mathrm{LEXP}(\mathbf{V}) \geq F_{\max}$ we replace the conditions (3) with more accurate ones. $\square$

Lemma 2. *At the maximum value of the function* $F(\alpha_1, \alpha_2, \dots, \alpha_{s+1})$ *all inequalities in the third condition of* (3) *are strict, i.e.,* $\alpha_1 > \alpha_2 > \cdots > \alpha_{s+1} > 0$.

Proof. Instead of the function $F(\alpha_1, \alpha_2, \dots, \alpha_{s+1})$ we can consider its logarithm

$$\ln(F(\alpha_1, \alpha_2, \dots, \alpha_{s+1})) = -\alpha_1 \cdot \ln(\alpha_1) - \alpha_2 \cdot \ln(\alpha_2) - \cdots - \alpha_{s+1} \cdot \ln(\alpha_{s+1}).$$

We shall show that if some of the inequalities $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_{s+1} \geq 0$ in the third condition of (3) are equalities, then some of the variables α_i can be changed with saving the other conditions of (3) in such a way that the number of the equations in the third line will reduce, and the value of F (as well as $\ln(F)$) will increase. Thus after a finite number of steps we can remove all of the equalities in the third line of (3) ensuring an increase in the value of the function. Really, we can consider the few possible cases.

CASE 1 (the leftmost and centre). In the third line there is a fragment of the form:

$$\alpha_{l-1} > \alpha_l = \cdots = \alpha_k > \alpha_{k+1} \geq 0, \text{ where } 1 < l < k < s+1 \text{ and } \alpha_0 := 1, \text{ if } l = 1.$$

Change the variables: $(\widetilde{\alpha}_l, \widetilde{\alpha}_k, \widetilde{\alpha}_{k+1}) = (\alpha_l + \beta, \alpha_k - 2\beta, \alpha_{k+1} + \beta)$, and leave the other variables unchanged. If $\beta > 0$ is small, the variation respects (3). We consider $F = F(\beta)$ as a function of β. We can verify that $(\ln(F(\beta)))'_{\beta=0+} = \ln(\alpha_k/\alpha_{k+1}) > 0$ and for small positive β the function $F(\beta)$ increases (it is true for $\alpha_{k+1} = 0$, when this derivative tends to $+\infty$).

CASE 2 (right extreme trivial). There is a fragment of the form:

$$\alpha_{k-2} > \alpha_{k-1} > \alpha_k > \alpha_{k+1} = \cdots = \alpha_{s+1} = 0, \text{ where } k \geq 2 \text{ and } \alpha_0 := 1, \text{ if } k = 2.$$

Now the variation $(\widetilde{\alpha}_{k-1}, \widetilde{\alpha}_k, \widetilde{\alpha}_{k+1}) = (\alpha_{k-1} + \beta, \alpha_k - 2\beta, \alpha_{k+1} + \beta)$ respects (3) and for small positive β increases both $F(\beta)$ and the number of strict inequalities in the third line of (3).

CASE 3 (the rightmost positive). There is a fragment of the considered line of the form:

$$\alpha_{k-2} \geq \alpha_{k-1} > \alpha_k = \cdots = \alpha_{s+1} > 0, \text{ where } 2 \leq k < s+1 \text{ and } \alpha_0 := 1, \text{ if } k = 2.$$

Change the value of the variables: $(\widetilde{\alpha}_{k-1}, \widetilde{\alpha}_k, \widetilde{\alpha}_{s+1}) = (\alpha_{k-1} - \beta, \alpha_k + 2\beta, \alpha_{s+1} - \beta)$, respecting (3). Then the value of the function $F(\beta)$ increases for small positive β.

There are still two cases to be considered:

- $\alpha_1 > \alpha_2 = \cdots = \alpha_{s+1} = 0$;
- $\alpha_1 = \alpha_2 = \cdots = \alpha_{s+1} > 0$.

In the former case, if $s \geq 2$, the function does not achieve its maximum. In the latter case we have $\alpha_i = 1/s$ and (3) implies the restriction $s \leq 2$. So, Lemma 2 is proved. □

Corollary 1. *The maximum of* $F(\alpha_1, \ldots, \alpha_{s+1})$ *on the domain satisfying* (3) *is attained only at* $\alpha_{s+1} \neq 0$. *Therefore these maximums strictly increase on* s:

$$\max_{(3)} F(\alpha_1, \ldots, \alpha_s, \alpha_{s+1}) > \max_{(3), \alpha_{s+1}=0} F(\alpha_1, \ldots, \alpha_s, \alpha_{s+1}) = \max_{(3)} F(\alpha_1, \ldots, \alpha_s).$$

Lemma 3. *When $s > 2$ condition (3) can be replaced by more exact*

$$(4) \qquad \begin{cases} \alpha_1 + \alpha_2 + \cdots + \alpha_{s+1} = 1 \\ \displaystyle\sum_{i=1}^{s+1} (2-i)\cdot\alpha_i = 0 \\ \alpha_1 > \alpha_2 > \cdots > \alpha_{s+1} > 0. \end{cases}$$

Proof. First, we find the maximum of $F(\alpha_1, \ldots, \alpha_{s+1}) = \alpha_1^{-\alpha_1} \cdot \cdots \cdot \alpha_{s+1}^{-\alpha_{s+1}}$ with the only condition $\alpha_1 + \cdots + \alpha_{s+1} = 1$. It becomes clear that the point $\overrightarrow{\alpha} = (1/s+1, \ldots, 1/s+1)$ is the only possible strict local maximum of F for $s+1$. When some α_i vanishes on the boundary of the domain, by the continuity, the corresponding factors become unit, and F decreases. If any of the α_i grows infinitely, the function F converges to zero. Hence, the obtained stationary point is a strict global maximum of F. For $s > 3$ the domain $\sum_{i=1}^{s+1}(i-2)\cdot\alpha_i < 0$ does not include the extremal point $(1/s+1, \ldots, 1/s+1)$. Thus, the local maximum of F for conditions (3) belongs to the border of the domain stated above and satisfies (4). Lemma 3 is proved. $\square$

Let us return to the proof of $\mathrm{LEXP}(\mathbf{V}) \geq F_{\max}$. We construct the sequence $\overrightarrow{\alpha}^{(s)}$ with rational components in the domain T, $s = 1, 2, \ldots$, so that $\lim_{s\to\infty} \overrightarrow{\alpha}^{(s)} = \overrightarrow{\alpha}^{(0)}$, $\lim_{s\to\infty} F(\overrightarrow{\alpha}^{(s)}) = F_{\max}$, and $m_{\lambda^{(s)}(t)} \neq 0$ in (1) for any positive integers s and t.

Let $\lambda = (\alpha_1 n, \alpha_2 n, \ldots, \alpha_{s+1} n) \vdash n$, where n is the common denominator of α_i. For all positive integers t let $\lambda(t) = (\alpha_1 nt, \alpha_2 nt, \ldots, \alpha_{s+1} nt) \vdash nt$. Now we shall take free generators $x_1, \ldots, x_{s+1}, x_{01}$, and x_{02} of the relatively free algebra $F(\mathbf{V})$. Recall that the capital letter denotes an appropriate inner derivation of the algebra. Denote

$$R_k = [\ldots[\overline{X}_1, \underbrace{X_{01}], \ldots X_{01}}_{k-2}][\ldots[\overline{X}_2, \underbrace{X_{01}], \ldots X_{01}}_{k-3}] \ldots [\overline{X}_{k-2}, \overline{X}_{01}]\overline{X}_{k-1},$$

where $k = 3, \ldots, s+1$. So, $R_3 = [\overline{X}_1, \overline{X}_{01}]\overline{X}_2$ and for example

$$R_4 = [[\overline{X}_1, X_{01}], X_{01}][\overline{X}_2, \overline{X}_{01}]\overline{X}_3.$$

Let also $R_1 = X_{01}$, $R_2 = X_1$.

Consider the following element of the relatively free algebra $F(\mathbf{V})$:

$$g_t = x_{02} R_1^{(\alpha_1 - \sum_{i=2}^{s+1}(i-2)\cdot\alpha_i)nt} R_2^{(\alpha_2-\alpha_3)nt} \cdots R_k^{(\alpha_k - \alpha_{k+1})nt} \cdots R_{s+1}^{\alpha_{s+1}nt}.$$

In the element g_t we also alternate the variable X_1 from R_2 with X_{01} from R_k, where $4 \leq k \leq s+1$. According to Lemma 3, $\sum\limits_{i=1}^{s+1}(2-i)\cdot\alpha_i = 0$, thus,

$$g_t = x_{02} R_2^{(\alpha_2-\alpha_3)nt} \cdots R_k^{(\alpha_k-\alpha_{k+1})nt} \cdots R_{s+1}^{\alpha_{s+1}nt}.$$

The degree of g_t is $nt + 1$.

Let f_t be the complete linearization of the element g_t, and let R_t be the S_{nt+1}-submodule of $P_{nt+1}(\mathbf{V})$, generated by f_t. The element g_t contains $\alpha_{s+1}nt$ alternating sets of $s+1$ variables $\{x_{01}, x_1, x_2, \ldots, x_s\}$ in each, and $(\alpha_i - \alpha_{i+1})nt$ alternating sets of i variables $\{x_{01}, x_1, \ldots, x_{i-1}\}$ in each, where $i = 2, \ldots, s$. All other variables, except x_{02}, which are not included in alternating sets, are equal to the same x_{01}. Therefore, the decomposition of the module R_t into a direct sum of irreducible components has only modules indexed by Young diagrams which contain a subdiagram corresponding to the partition $\lambda(t)$.

We shall prove that at least one of these irreducible submodules is not zero in the module of multilinear polynomials $P_{nt+1}(\mathbf{V})$. Consider the elements $h_i = x_{02}R_k$, $k = 2, \ldots, s+1$. Make the following substitution in $h_2, \ldots, h_{s+1}$: $x_{02} = z_1 Z_s^m$, $m > 0$, $x_1 = z_s$, $x_2 = z_{s-1}$, $\ldots$, $x_s = z_1$, $x_{01} = d$. If two elements z_i, z_j in the summation fall both into the commutator bracket, then such term is zero, because M is a metabelian ideal of L_s. Hence only one from the $k!$ terms is not equal to zero and the result of this substitution is equal to $\pm z_1 Z_s^{k-1+m}$.

Thus, if in the element g_t we make the following substitution of elements of L: $x_{02} = z_1 Z^m$, $x_1 = z_s$, $x_2 = z_{s-1}$, $\ldots$, $x_s = z_1$, $x_{01} = d$, the result of the substitution is not zero.

From these inequalities we obtain that $\mathrm{LEXP}(\mathbf{V}) = \mathrm{HEXP}(\mathbf{V}) = F_{\max}$. The problem reduces to find the maximum of the function $F(\alpha_1, \ldots, \alpha_{s+1})$.

Lemma 4. *For $s \geq 3$*

$$\max_{(4)} F(\alpha_1, \ldots, \alpha_{s+1}) = (s+1) \cdot q^{1-s}/(2-q),$$

where $q = q(s+1)$ is a root of the polynomial $P(x) = -x^s + x^{s-2} + 2x^{s-3} + \cdots + (s-2)x + (s-1)$.

Proof. The study of the Lagrangian of F subject to the conditions (4) gives $\alpha_i/\alpha_{i+1} = q$, $\alpha_i = q^{s+1-i} \cdot \alpha_{s+1}$ and $\alpha_{s+1} = (q-1)/(q^{s+1}-1) = (2-q)/(s+1)$. $\square$

Note that there are two other equations satisfied by $q(s+1)$

$$-x^{s+1} + \sum_{i=1}^{s} x^i - s + 1 = 0 \quad \text{and} \quad -x^{s+2} + 2x^{s+1} - sx + s - 1 = 0.$$

Lemma 5. *If $s \geq 2$, then:*

(1) *the equation $P(x) = 0$ has unique positive solution*;

(2) *$q(s+1)$ belongs to $[1, 2)$ and strictly increases with s*;

(3) $\lim\limits_{s \to +\infty} q(s+1) = 2$.

Proof. (1) Clearly, $x = 0$ is not a solution. We rewrite the equation in the form

$$(1/x)^2 + 2(1/x)^3 + \cdots + (s-2) \cdot (1/x)^{s-1} + (s-1) \cdot (1/x)^s = 1.$$

The left side is strictly increasing for positive $1/x$ from $0+$ to $+\infty$, so the equality is realized in a unique $x = q(s+1)$.

(2) We have $P(1) \geq 0$, but $P(2) < 0$, therefore $q(s+1) \in [1, 2)$. As we have already shown $y = q(s+1)^{-1}$ satisfies: $y^2 + 2y^3 + \cdots + (s-2) \cdot y^{s-1} + (s-1) \cdot y^s = 1$, also $z = q(s+2)^{-1}$ complies with $z^2 + 2z^3 + \cdots + (s-2) \cdot z^{s-1} + (s-1) \cdot z^s + s \cdot z^{s+1} = 1$. If for some s the inequality $1/q(s+2) \geq 1/q(s+1)$ holds, then the expression of z exceeds the expression of y, but both are equal to 1. Therefore $1/q(s+2) < 1/q(s+1)$ and $q(s+2) > q(s+1)$ for all $s \geq 2$.

(3) The sequence $q(s+1) < 2$ grows, hence $q(+\infty) = \lim\limits_{s \to +\infty} q(s+1) \in (1, 2]$ exists. Then $1/q(+\infty) \in [0.5, 1)$ is a solution of the equation

$$1 = x^2 + 2x^3 + \cdots + (s-1) \cdot x^s + \cdots = x^2/(1-x)^2.$$

Consequently, $1/q(+\infty) = 0.5$ and $q(+\infty) = 2$. Lemma 5 is proved. $\square$

Let $F(s+1) = \max_{(4)} F(\alpha_1, \ldots, \alpha_{s+1})$ and let $\alpha_i(s+1)$ be the corresponding optimal values of the variables for $1 \leq i \leq s+1$.

Lemma 6.

(1) $\lim\limits_{s \to +\infty} F(s+1) = 4$;

(2) $3 = F(3) < \cdots < F(s+1) < F(s+2) < \cdots < 4$;

(3) $\lim\limits_{s \to +\infty} \alpha_i(s+1) = 2^{-i}$.

Proof. (1) According to the proof of Lemma 4,

$$\alpha_{s+1}(s+1) = \frac{q(s+1) - 1}{q(s+1)^{s+1} - 1}, \quad F(s+1) = \frac{q(s+1)^{s+1} - 1}{q(s+1)^{s-1} \cdot (q(s+1) - 1)}.$$

Then

$$\lim_{s \to +\infty} F(s+1) = \lim_{s \to +\infty} \frac{q(s+1)^2}{q(s+1) - 1} \cdot \frac{q(s+1)^{s+1} - 1}{q(s+1)^{s+1}} = \lim_{q \to 2} \frac{q^2}{q - 1} = 4.$$

(2) By Corollary 1, the sequence $F(s+1)$ strictly increases, therefore it holds

$$\sup_{s \geq 0} F(s+1) = \lim_{s \to +\infty} F(s+1) = 4.$$

(3) Using Lemma 4 again, we deduce

$$\lim_{s \to +\infty} \alpha_i(s+1) = \lim_{s \to +\infty} \frac{q(s+1) - 1}{q(s+1)^i} \cdot \frac{q(s+1)^{s+1}}{q(s+1)^{s+1} - 1} = \lim_{q \to 2} \frac{q - 1}{q^i} = 2^{-i}.$$

Thus Lemma 6 and the theorem are proved. $\square$

REFERENCES

[1] Regev A. Bull. Amer. Math. Soc., **77**, 1971, No 6, 1067–1069.
[2] Giambruno A., M. Zaicev. Adv. Math., **142**, 1999, 221–243.
[3] Giambruno A., S. P. Mishchenko, M. V. Zaicev. Adv. Math., **217**, 2008, 1027–1052.
[4] Mishchenko S. P., M. V. Zaicev. J. Math. Sci. (New York), **93**, 1999, No 6, 977–982.
[5] Verevkin A. B., M. V. Zaicev, S. P. Mishchenko. Vestnik Moskov. Univ. Ser. I Mat. Mekh., 2011, No 2, 36–39 (in Russian). Translation: Moscow Univ. Math. Bull., **66**, 2011, 86–89.
[6] Mishchenko S. S. Vestnik Moskov. Univ. Ser. I Mat. Mekh., 2011, No 6, 44–47 (in Russian). Translation: Moscow Univ. Math. Bull., **66**, 2011, 264–266.
[7] Giambruno A., M. Zaicev. Polynomial Identities and Asymptotic Methods, Mathematical Surveys and Monographs, **122**, AMS, Providence, RI, 2005.
[8] Mishchenko S. P. Vestnik Moskov. Univ. Ser. I Mat. Mekh., 1993, No 1, 90–91 (in Russian). Translation: Moscow Univ. Math. Bull., **48**, 1993, 63–64.
[9] Zaicev M. V., S. P. Mishchenko. Algebra i Logika, **38**, 1999, 161–175 (in Russian). Translation: Algebra Logic, **38**, 1999, 84–92.

Department of Algebra and Geometric Computations
Ulyanovsk State University
Ulyanovsk 432970, Russia
e-mails: malyusheva_o_a@mail.ru
mishchenkosp@mail.ru
abverevkin@gmail.com