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ON LIE ALGEBRAS WITH EXPONENTIAL GROWTH OF THE CODIMENSIONS

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Abstract. We study the asymptotic behaviour of the codimension sequence of varieties of Lie algebras variety over a field of characteristic zero. We construct an infinite series of such varieties with different fractional exponents. This extends the special cases known before.

1. Introduction. There are different ways to investigate linear algebras. Instead to study an individual algebra one can analyze a class of algebras with similar properties. As such a class one may consider the variety, defined as the class of algebras satisfying fixed polynomial identities. One can specify the identities explicitly, for example – the identity of commutativity, or implicitly – studying the variety generated by a fixed algebra A. All algebras of the variety satisfy the identities of A, even if they are not described explicitly. In this case, the chosen algebra A is said to be a support of the variety. For a support of

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variety we can take its relatively free algebra of countable rank. This is the countable generated algebra having no relations, except of the fixed identities and their consequences. It is known that the variety is a class of algebras stable under subquotients and Cartesian products.

In this paper we consider some numerical characteristics of varieties of Lie algebras over a field of characteristic zero. Here we have constructed an infinite series of varieties with different fractional exponents. This extends the special cases known before. In the sequel we shall follow the work [1], [4], where the necessary background was stated.

In the middle of the XX century it was discovered that over a field of characteristic zero any polynomial identity is equivalent to a system of multilinear identities [7]. So, any information about the variety $\mathbf{V}$ is contained in properties of the sequence of vector spaces $P_n(\mathbf{V})$, $n = 1, 2, \ldots$, of the multilinear elements of the relatively free algebra.

An important characteristic of the variety is the codimension sequence $c_n(\mathbf{V}) = \dim P_n(\mathbf{V})$, $n = 1, 2, \ldots$. The growth of this sequence determines the growth of the variety. There are varieties with polynomial, exponential, intermediate (between polynomial and exponential) and overexponential growth. Unlike the associative case, for Lie algebras the sequence $c_n(\mathbf{V})$ is already not necessarily exponentially limited. The existence of varieties of Lie algebras with overexponential growth was established in [16]. In [12], [13] it was introduced a scale to measure the overexponential growth for varieties of polynilpotent Lie algebras.

In the case of exponential growth the sequence $\sqrt[n]{c_n(\mathbf{V})}$ is bounded and has lower and upper limits, known as the lower and the upper exponents of the variety, respectively. When they are equal, the limit of the sequence exists and is called the exponent of the variety. For example, for the variety of associative algebras generated by the Grassmann algebra the codimensions sequence has the form $c_n(\mathbf{V}) = 2^{n-1}$ (see [4], theorem 4.1.8) and the exponent is 2.

The question for the existence of the exponent of an arbitrary variety of exponential growth is a very challenging and interesting problem. Up till now, the proofs for the existence of the exponent and its evaluation are related with big difficulties. Another interesting problem is to find varieties with integer exponents and, in the contrary, examples of varieties with fractional exponents.

In the 1980's S. A. Amitsur conjectured that any associative algebra satisfying a polynomial identity has a non-negative integer exponent. His conjecture was proved in the work [3].

The first example of a variety of Lie algebras with non-integer exponent was constructed in [19]. Then in [15] the existence of its exponent was proved and its value was calculated. The integrity of the exponent of a variety of Lie algebras generated by a finite dimensional algebra was proved in [17]. For Lie algebras with nilpotent commutator the same was established in [10]. In general, it was shown in [2] that for any real number $\alpha > 1$ there exists a linear algebra A_α whihc generates a variety with exponent equal to α. Note, that a similar result for Lie algebras over a countable field would mean the affirmative answer to the unsolved problem of the existence of a Lie algebra without finite basis of its polynomial identities.

Another example of a variety of Lie algebras with fractional exponent was constructed in [11]. It is generated by the infinite dimensional simple algebra of Cartan type W_2, in other words – by the Lie algebra of the vector fields on the plane.

The latest known result in this direction is our work [8], where we have announced and briefly justified the existence of a discrete series of varieties of Lie algebras with different fractional exponents. In the present paper we give the detailed exposition of the result.

2. Main definitions and notations. Let Φ be a field of characteristic zero. We shall use left-normed arranging in the products, omitting the parentheses, for example, $(ab)c \equiv abc$.

Let $F(X)$ be a free algebra, generated by a countable set $X = \{x_1, x_2, \dots\}$. We shall write the identities as equalities in $F(X)$

$$f(x_1, \dots, x_n) \equiv 0 \quad \text{or} \quad f(x_1, \dots, x_n) \equiv g(x_1, \dots, x_n),$$

where we use the equivalence instead of the ordinary equality.

Let A be any Lie algebra over Φ. Recall that $f(x_1, \dots, x_n)$ is a polynomial identity of A if $f(a_1, \dots, a_n) = 0$, for any $a_1, \dots, a_n \in A$. Let $\mathrm{Id}(A)$ denote the set of all identities of A. Then $\mathrm{Id}(A)$ is an ideal in $F(X)$ invariant under substitutions, i.e., $f(g_1, \dots, g_n) \in \mathrm{Id}(A)$ for any $g_1, \dots, g_n \in F(X)$ and $f(x_1, \dots, x_n) \in \mathrm{Id}(A)$. Such ideals are called fully invariant ideals or T-ideals.

For any $n \geq 1$, P_n denotes the vector subspace of $F(X)$ of multilinear polynomials in $x_1, \dots, x_n$. Hence $P_n \cap \mathrm{Id}(A)$ is the vector space of multilinear identities of A of degree n. In characteristic zero the sequence of vector spaces $P_n \cap \mathrm{Id}(A)$, $n = 1, 2, \dots$, completely determines $\mathrm{Id}(A)$ and we define $P_n(A) = P_n/(P_n \cap \mathrm{Id}(A))$, which in some sense corresponds to the non-identities of A of n-th degree.

The important characteristic of the identities of A is the codimensions sequence

$$c_n(A) = \dim P_n(A),\ n = 1, 2, \ldots.$$

If the algebra B satisfied all identities of the algebra A, then we have $\mathrm{Id}(B) \supseteq \mathrm{Id}(A)$ and $c_n(B) \le c_n(A)$. So, $c_n(A) = n!$ when A is the free associative algebra, and $c_n(A) = (n-1)!$ when A is the free Lie algebra. The fastest growing sequence is realized when A satisfies no identities. Then $c_n(A) = \binom{2n-2}{n-1} \cdot (n-1)!$, where $\binom{2n-2}{n-1}/n$ is the n-th Catalan number, counting all arrangements of parentheses on a word of length n. Notice that asymptotically $\binom{2n-2}{n-1}/n \simeq 4^n$.

There is a wide class of algebras with exponentially bounded growth of the codimensions, i.e., $c_n(A) \le a^n$, for all n, with a a real number. Then the sequence $\sqrt[n]{c_n(A)}$, $n = 1, 2, \ldots$, is bounded: $0 \le \sqrt[n]{c_n(A)} \le a$. Its lower and upper limits exist and are called the lower and upper exponent of A

$$\underline{\exp}(A) = \liminf_{n\to\infty} \sqrt[n]{c_n(A)}, \quad \overline{\exp}(A) = \limsup_{n\to\infty} \sqrt[n]{c_n(A)}.$$

If the limit of the sequence $\sqrt[n]{c_n(A)}$ exists, then we call it the PI-exponent (polynomial identity exponent) or just the exponent of A:

$$\exp(A) = \underline{\exp}(A) = \overline{\exp}(A).$$

As we said, one of the main problems in the theory of codimensions is the existence of the PI-exponent of A, when $c_n(A)$ is exponentially bounded.

There are other numerical characteristics of T-ideals which are closely related to the action of the symmetric group S_n on P_n. Recall that if $f = f(x_1, \ldots, x_n) \in P_n$ and $\sigma \in S_n$, then

$$\sigma f(x_1, \ldots, x_n) = f(x_{\sigma(1)}, \ldots, x_{\sigma(n)}).$$

This action defines a representation of the group S_n in the space $P_n(A)$. Since $\mathrm{Id}(A)$ is invariant under endomorphisms, it follows that $P_n \cap \mathrm{Id}(A)$ is a ΦS_n-submodule of P_n and $P_n(A) = P_n/(P_n \cap \mathrm{Id}(A))$ is also a ΦS_n-module. Explicitly, two multilinear identities $f \equiv 0$ and $g \equiv 0$ are equivalent if and only if $\Phi S_n f = \Phi S_n g$.

Since $\operatorname{char} F = 0$, the ΦS_n-module $P_n(A)$ is completely reducible and can be decomposed into a direct sum of irreducible submodules. The number of irreducible summands, i.e., the length of the ΦS_n-module $P_n(A)$ is called the n-th colength of A, and is denoted by $l_n(A)$. Hence a new numerical invariant of $\mathrm{Id}(A)$ is given by the colength sequence $l_n(A),\ n = 1, 2, \ldots$.

The partitions $\lambda = (\lambda_1, \ldots, \lambda_k)$ of n are in one-to-one correspondence with the set of nonequivalent irreducible representations of S_n, that are Specht modules S^λ.

Finite-dimensional representations of S_n are determined by their characters (the traces of the corresponding linear operators) which are central functions on S_n. The characters of irreducible representations of S_n form an orthogonal basis of the space of central functions and can be normalized if the field Φ is algebraically closed (we refer the reader to [5] for an account of the representation theory of the symmetric group).

For describing the decomposition of an S_n-module M into irreducibles it is convenient to use characters. Hence if χ_λ denotes the character of S^λ, we write

$$\chi(M) = \sum_{\lambda \vdash n} m_\lambda \chi_\lambda,$$

where $\chi(M)$ is the character of M and m_λ is the multiplicity of χ_λ in $\chi(M)$.

The S_n-character $\chi_n(A)$ of $P_n(A)$ is said to be the n-th cocharacter of A. Then

$$\chi_n(A) = \chi(P_n(A)) = \sum_{\lambda \vdash n} m_\lambda \chi_\lambda, \tag{1}$$

Then clearly

$$l_n(A) = \sum_{\lambda \vdash n} m_\lambda, \quad c_n(A) = \sum_{\lambda \vdash n} m_\lambda d_\lambda,$$

where $d_\lambda = \chi_\lambda(e) = \deg \chi_\lambda$ is the degree of the irreducible character χ_λ.

The capital letter Y will denote the inner derivation $\operatorname{ad} y$ of the Lie algebra A: $\operatorname{ad} y(x) = xY = xy$, where $x,\ y \in A$. The bar or the tilde are used to denote the alternation of the elements in the expression. For example,

$$x_0 \bar{X}_1 \tilde{Y}_1 \bar{X}_2 \tilde{Y}_2 \tilde{Y}_3 \bar{X}_3 \bar{X}_4 = \sum_{p \in S_4,\ q \in S_3} (-1)^{p+q} x_0 x_{p(1)} y_{q(1)} x_{p(2)} y_{q(2)} y_{q(3)} x_{p(3)} x_{p(4)},$$

where $(-1)^r$ is the parity of the permutation $r \in S_n$. Note the equality

$$x_0 \ldots \bar{X}_1 \ldots \bar{X}_2 \ldots \bar{X}_m = (-1)^p x_0 \ldots \bar{X}_{p(1)} \ldots \bar{X}_{p(2)} \ldots \bar{X}_{p(m)}, \quad p \in S_m.$$

The above is explained on the following examples:

$$y_1 \bar{X}_1 [\bar{X}_2, \bar{Y}] = 2(y_1 x_1 x_2 y + y_1 x_2 y x_1 + y_1 y x_1 x_2 - y_1 x_1 y x_2 - y_1 y x_2 x_1 - y_1 x_2 x_1 y),$$

$$\bar{X}_1[\bar{X}_2,\bar{X}_3][[\bar{X}_4,\bar{X}_5],Y]=\sum_{p\in S_5}(-1)^p X_{p(1)}[X_{p(2)},X_{p(3)}][[X_{p(4)},X_{p(5)}],Y].$$

Here the notation for the commutator for the associative composition of linear operators is used. We also have applied the identity $x(yz) \equiv xyz - xzy$ which holds for any Lie algebra.

3. Main result. In this section we construct an infinite series of varieties of Lie algebras with different fractional PI-exponents. It is the main result of this paper.

Let $\mathbf{A}^2$ be the variety of all metabelian Lie algebras determined by the identity

$$(x_1x_2)(x_3x_4) \equiv 0,$$

and let $M_{s-1} = F_{s-1}(\mathbf{A}^2)$, $s = 3, 4, \ldots$, denote the relatively free algebra of this variety over the set of free generators $\{z_1, z_2, \ldots, z_{s-1}\}$. Consider the linear transformation d of the vector space $\langle z_1, z_2, \ldots, z_{s-1}\rangle$ spanned by $z_1, z_2, \ldots, z_{s-1}$, defined by the rule $z_i d = z_{i+1}$, $i = 1, 2, \ldots, s-2$, $z_{s-1}d = 0$. Then d can be extended to a derivation of the algebra M_{s-1}, denoted by the same letter. Let $\langle d\rangle$ be the one-dimensional Lie algebra generated by d with zero-multiplication. We may construct the semidirect product $L_s = M_{s-1} \leftthreetimes \langle d\rangle$. The variety generated by the Lie algebra L_s is denoted by $\mathrm{var}(L_s)$, $s = 3, 4, \ldots$.

Theorem. *The following strict inequalities hold for the exponent of the codimension sequences of the algebras L_s*

$$3 = \exp(L_3) < \cdots < \exp(L_s) < \exp(L_{s+1}) < \cdots < 4, \ \textit{where } s = 4, 5, \ldots.$$

The proof of the theorem will require the following statement.

Lemma 1. *If the multiplicity m_λ from* (1) *is different from zero for the Lie algebra L_s and then the following inequalities hold for the partition $\lambda = (\lambda_1, \ldots, \lambda_k) \vdash n$*

$$\sum_{i>s}\lambda_i < 2, \quad \sum_{i=1}^{s}(2-i)\cdot\lambda_i + (s-1)\cdot(s-2) \geq 0.$$

P r o o f. Let us establish the first inequality. Assume the contrary and suppose that $\lambda \vdash n$ has more than two cells outside of the first s rows in the corresponding Young diagram. Let $\lambda'_1, \dots, \lambda'_{l(\lambda)}$ be the heights of the columns of this diagram. Consider an element f of $P_n(L_s)$ which generates an irreducible S_n-module corresponding to $\lambda \vdash n$. According to [9], the element f is equal to a linear combination of summands on λ_1 skew-symmetric sets with λ'_i variables in the i-th set. But any multilinear Lie polynomial, containing either $s+2$ skew-symmetric variables or two sets with $s+1$ skew-symmetric variables, vanishes in algebra L_s, because the algebra contains the abelian ideal M^2_{s-1} of codimension s.

Really, let the multilinear polynomial contain $s+2$ or more skew-symmetric variables. It is sufficient to check, that it vanishes on the basis of the algebra, replacing the skew-symmetric variables by pairwise different elements of the basis. We shall construct the basis of L_s extending the basis of the abelian ideal M^2_{s-1}. But modulo this ideal there exist only s linearly independent elements and we can choose the following basis elements $\{d, z_1, z_2, \dots, z_{s-1}\}$. Hence we shall replace the skew-symmetric variables by at least two elements from M^2_{s-1}. Therefore any monomial from our multilinear Lie polynomial vanishes, because it can be presented as a product of two elements of the ideal M^2_{s-1} with zero multiplication.

Now, let the multilinear polynomial contains two different skew-symmetric sets of $s+1$ variables. After the replacement of the skew-symmetric variables by basis elements each skew-symmetric set gives rise to an element of M^2_{s-1}. Again, every monomial from our Lie polynomial will vanish.

In this way, the polynomial f corresponding of the partition λ vanishes under any substitution by elements from L_s and the multiplicity m_λ is equal to zero.

Let us prove the second inequality. Consider a partition $\lambda = (\lambda_1, \lambda_2, \dots) \vdash n$ such that $\sum\limits_{i \le s} \lambda_i \ge n-1$ and $\sum\limits_{i \le s} (2-i)\cdot\lambda_i + (s-1)\cdot(s-2) < 0$. We shall show that this partition determines a polynomial identity of the algebra L_s. It is sufficient to prove that any multilinear polynomial f depending upon $l = \lambda_1$ skew-symmetric sets of variables with $\lambda'_1, \dots, \lambda'_l$, elements, respectively, takes only zero value in L_s.

Again, we fix a basis of the abelian ideal M^2_{s-1} and extend it to a basis of L_s by the elements $d, z_1, z_2, \dots, z_{s-1}$. We shall replace the variables of f with some of the basis elements. The element d can enter not more than once in each skew-symmetric set, otherwise f vanish. Let us identify the variables in f which we replace by d, and denote them by b. The other variables will be denoted by $y_1, \dots, y_k$. Taking into account that d is a derivation we can rewrite the polynomial f as a linear combination of the following products

$$(y_{s_1} b^{\alpha_1})(y_{s_2} b^{\alpha_2}) \dots (y_{s_k} b^{\alpha_k}), \tag{2}$$

where $\alpha_1, \dots, \alpha_k \geq 0$. Note also, that $\alpha_1 + \dots + \alpha_k \leq \lambda_1$.

The elements $y_i b^\alpha$, $\alpha \geq 0$, may be considered as new variables. Although the polynomial f may be not multilinear in these variables, it can be written as a sum $f = f_1 + \dots + f_m$, such that each f_i is a multilinear polynomial in some of the new variables. If f is skew-symmetric in y_1 and y_2, then those elements from $f_1, \dots, f_m$, which depend on $y_1 b, y_2 b$ are skew-symmetric in these variables too. Similarly the skew-symmetry in $y_1 b^j, y_2 b^j$, where $j = 2, 3, \dots$ will be preserved.

Now shall we prove, that each component $f_1, \dots, f_m$ takes zero value. For this purpose we write for example f_1 as a linear combination of elements of the form (2), and then fix the indices s_1, s_2. We shall show that the partial sum f_1^1 of f_1 for these fixed s_1, s_2 at the first two positions equals zero. Really, let f be skew-symmetric in $y_1, y_2, \dots, y_r$, where $1, 2, \dots, r \neq s_1, s_2$ and let f_1^1 depend on y_1, y_2 or on $y_1 b^j, y_2 b^j$ for some $j = 1, 2, \dots$. Then the evaluations of $y_{s_i} b^{\alpha_i}$, $i = 1, \dots, k$, belong to the metabelian ideal M_{s-1} of algebra L_s. We shall use the identity $x_1 x_2 x_{\sigma(3)} \dots x_{\sigma(k)} \equiv x_1 x_2 x_3 \dots x_k$ which holds in M_{s-1} for any permutation σ of $3, \dots, k$. Hence the evaluations of $y_1 b^j, y_2 b^j$ commute and are skew-symmetric in the same time. This implies that the component f_1^1 of f_1 takes value zero.

In other words, if f_1^1 depends on $y_1 b^{\alpha_1}, y_2 b^{\alpha_2}, \dots, y_i b^{\alpha_i}$ and takes a non-zero value, then all $\alpha_1, \alpha_2, \dots, \alpha_i$ are pairwise different and $\alpha_1 + \alpha_2 + \dots + \alpha_i \geq i(i-1)/2$, $i = 2, 3, \dots$. Recall, that the original polynomial f depends on λ_s skew-symmetric sets of cardinality $\leq s$ ($> s$, when $\lambda_{s+1} = 1$) and depends on $\lambda_i - \lambda_{i+1}$ skew-symmetric sets of cardinality i, where $i = 3, \dots, s-1$. The variables y_{s_1} and y_{s_2} are used not more than twice in these sets and it is possible to substitute by d only one variable from each set. So we have at most $\lambda_s - 2$ skew-symmetric sets with s elements and $\lambda_i - \lambda_{i+1}$ sets with $i-1$ elements, where $i = 3, \dots, s-1$. We have shown above that f_1^1 may take non-zero values only if the following condition holds

$$\alpha_1 + \dots + \alpha_k \geq \sum_{i=3}^{s-1} \frac{(i-1)(i-2)}{2} \cdot (\lambda_i - \lambda_{i+1}) + \frac{(s-1)(s-2)}{2} \cdot (\lambda_s - 2).$$

But $\alpha_1 + \dots + \alpha_k \leq \lambda_1$ implies the inequality $\displaystyle\sum_{i=1}^{s} (2-i) \cdot \lambda_i + (s-1) \cdot (s-2) \geq 0$. It means that if $\displaystyle\sum_{i=1}^{s} (2-i) \cdot \lambda_i + (s-1) \cdot (s-2) < 0$, then f_1^1, f_1 and hence f take zero values only. Our Lemma 1 is proved. □

We return to the proof of the main theorem. Let M_n be set of partitions of n which, according to Lemma 1, may have non-zero multiplicities. It consists of partitions $\lambda \vdash n$ which satisfy the inequalities

$$\sum_{i>s} \lambda_i \leq 1, \quad \sum_{i\leq s} (2-i)\cdot\lambda_i + (s-1)\cdot(s-2) \geq 0.$$

Denote by T_n the set of partitions satisfying the following conditions

$$\left\{\begin{array}{rcl} \lambda_1+\lambda_2+\cdots+\lambda_s & = & n \\ \sum_{i=1}^{s}(2-i)\cdot\lambda_i & \geq & 0 \\ \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_s & \geq & 0. \end{array}\right. \tag{3}$$

For the non-negative real variables $\alpha_1, \alpha_2, \ldots, \alpha_s$ we shall define the function

$$F(\overrightarrow{\alpha}) = F(\alpha_1, \alpha_2, \ldots, \alpha_s) = \prod_{i=1}^{s} \alpha_i^{-\alpha_i}.$$

By continuity from the right we set $0^0 = 1$ for the zero values of the variables. Let T be the compact subset of the real space $\mathbb{R}^s$ defined by the conditions

$$\left\{\begin{array}{rcl} \alpha_1+\alpha_2+\cdots+\alpha_s & = & 1 \\ \sum_{i=1}^{s}(2-i)\cdot\alpha_i & \geq & 0 \\ \alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_s & \geq & 0. \end{array}\right. \tag{4}$$

The function $F(\overrightarrow{\alpha})$ is continuous and takes its maximal value on T at some $\overrightarrow{\alpha}^{(0)} \in T$

$$F_{\max} = F(\overrightarrow{\alpha}^{(0)}) = \max_{\overrightarrow{\alpha}\in T} F(\overrightarrow{\alpha}).$$

The maximal values of these functions depend on s. We denote them by $F(s)$, $s = 3, 4, \ldots$. Next we need the following property of $F(\alpha_1, \alpha_2, \ldots, \alpha_s)$.

Proposition. *If the function $F(\alpha_1, \alpha_2, \ldots, \alpha_s)$ reaches the maximal value on the compact subset T of $\mathbb{R}^s$, then the last variable α_s is different from zero and the second condition of (4) becomes an equality. Besides, $\lim_{s\to\infty} F(s) = 4$ and the following inequalities hold:*

$$3 = F(3) < \cdots < F(s) < F(s+1) < \cdots < 4.$$

The proof of the proposition will be given in the next section.

P r o o f o f t h e m a i n t h e o r e m. Lemma 1 asserts that if λ has at least two cells outside the first s rows or $\sum_{i\le s}(2-i)\cdot\lambda_i+(s-1)\cdot(s-2)<0$, then the multiplicity m_λ is equal to 0. In particular, the variety $\operatorname{var}(L_s)$ satisfies the system of Capelli identities in $m=s+2$ skew-symmetric variables of the form:

$$\sum_{p\in S_m}(-1)^p x_{p(1)}y_{11}\dots y_{1n_1}x_{p(2)}y_{21}\cdots y_{2n_2}x_{p(3)}\cdots x_{p(m)}\equiv 0,$$

where some n_i may be equal to zero. As it was proved in [18] if the Lie algebra A satisfies a system of Capelli identities, then the colength of the variety $l_n(A)$ is polynomially bounded. Therefore, the upper and lower the limits of the exponential functions L_s can be found by analyzing the dimensions d_λ of the irreducible modules of the symmetric group occurring in the decomposition of $P_n(L_s)$.

Let us recall the "hook-formula" ([6], p. 81) for the dimension of the irreducible representations S^λ of the symmetric group S_n corresponding to the partition $\lambda=(\lambda_1,\dots,\lambda_s)\vdash n$

$$d_\lambda=\dim S^\lambda=n!/\prod h_{ij}(\lambda).$$

Here the "length of the (i,j)-hook" is defined by $h_{ij}(\lambda)=(\lambda_i-j)+(\lambda'_j-i)+1$, where $\lambda'_j=|\{\lambda_i\ge j\}|$ is the j-th part of the dual partition λ' and the product in the denominator is taken over all natural pairs $\{(i,j)\,|\,i\le\lambda'_j,\ j\le\lambda_i\}$ from the "Young diagram" of the partition λ.

The hook-formula can be reduced to a more convenient form. It is proved in ([14], p. 28) that in the above notation we have the equality

$$\prod h_{ij}(\lambda)=\lambda_1!\lambda_2!\cdots\lambda_s!\cdot\prod_{i=1}^{s-1}\prod_{j=1}^{s-i}\frac{\lambda_i+j}{\lambda_i+j-\lambda_{i+j}}.$$

This implies the following connection between the dimensions of the representations and the generalized binomial coefficients:

$$(5)\quad d_\lambda=\frac{n!}{\lambda_1!\lambda_2!\cdots\lambda_s!}\prod_{i=1}^{s-1}\prod_{j=1}^{s-i}\frac{\lambda_i+j-\lambda_{i+j}}{\lambda_i+j}=\frac{n!}{\lambda_1!\lambda_2!\cdots\lambda_s!}\prod_{i=1}^{s-1}\prod_{j=1}^{s-i}\left(1-\frac{\lambda_{i+j}}{\lambda_i+j}\right).$$

For a positive integer t we define the partition $\lambda\cdot t=(\lambda_1 t,\lambda_2 t,\dots,\lambda_s t)\vdash nt$. Let $d_{\lambda\cdot t}$ be the dimension of the irreducible S_{nt}-modules $S^{\lambda\cdot t}$. Next, we use the following known fact.

Remark. Under the above conditions we have the equality

$$\lim_{t\to\infty} \sqrt[nt]{d_{\lambda\cdot t}} = F(\alpha_1, \alpha_2, \dots, \alpha_s), \text{ where } \quad \alpha_i = \lambda_i/n.$$

Really, by (5) the dimension $d_{\lambda\cdot t}$ and the generalized binomial coefficient differ by a rational factor of t of degree which does not depend on t. So, we can estimate the generalized binomial coefficient by the Stirling formula for the factorials

$$n! = \sqrt{2\pi n}\cdot\left(\frac{n}{e}\right)^n\cdot\left(1+O\left(\frac{1}{n}\right)\right).$$

For fixed n and $\lambda_i > 0$, and for $t \to \infty$ we obtain the equalities

$$\begin{aligned}\binom{nt}{\lambda_1 t, \dots, \lambda_s t} &= \left(\sqrt{2\pi nt}\left(\frac{nt}{e}\right)^{nt}\Big/\prod_{i=1}^{s}\sqrt{2\pi\lambda_i t}\left(\frac{\lambda_i t}{e}\right)^{\lambda_i t}\right)\cdot(1+O(1/t))\\ &= (2\pi t)^{\frac{1-s}{2}}\cdot\sqrt{\frac{n}{\lambda_1\cdots\lambda_s}}\cdot\frac{n^{nt}}{\lambda_1^{\lambda_1 t}\cdots\lambda_s^{\lambda_s t}}\cdot\left(\frac{e}{t}\right)^{(\lambda_1+\ldots+\lambda_s-n)t}\cdot(1+O(1/t))\\ &= (2\pi t)^{\frac{1-s}{2}}\cdot\sqrt{\frac{n}{\lambda_1\cdots\lambda_s}}\cdot\prod_{i=1}^{s}\left(\frac{n}{\lambda_i}\right)^{\lambda_i t}\cdot(1+O(1/t)).\end{aligned}$$

If $\lambda_s = 0$, the expression for the binomial coefficient is similar, but the products contain only the non-zero λ_i. Then we get the value of the desired limit

$$\binom{nt}{\lambda_1 t, \dots, \lambda_s t}^{\frac{1}{nt}} = F(\alpha_1, \alpha_2, \dots, \alpha_s)\cdot(1+o(1)) \to F(\alpha_1, \alpha_2, \dots, \alpha_s), \text{ for } t\to\infty.$$

Let the maximal value $F(s)$ of the function $F(\alpha_1, \dots, \alpha_s)$ on the compact set T be reached in a point $\overrightarrow{\alpha}^{(0)} = (\alpha_1^{(0)}, \dots, \alpha_s^{(0)})$. According to the proposition above, it satisfies the conditions: $\alpha_s^{(0)} > 0$ and $\sum_{i\le s}(2-i)\cdot\alpha_i^{(0)} = 0$. in We construct a sequence $\overrightarrow{\alpha}^{(j)}$, $j = 1, 2, \dots$, with similar conditions and rational components $\alpha_i^{(j)}$, such that

$$\lim_{j\to\infty}\overrightarrow{\alpha}^{(j)} = \overrightarrow{\alpha}^{(0)}, \quad \lim_{j\to\infty} F(\overrightarrow{\alpha}^{(j)}) = F(s).$$

Let $n^{(j)}$ be the common denominator of components of $\overrightarrow{\alpha}^{(j)}$. Define the partitions: $\lambda^{(j)} = \overrightarrow{\alpha}^{(j)}\cdot n^{(j)} \vdash n^{(j)}$ and $\lambda^{(j)}\cdot t \vdash n^{(j)}\cdot t$, for positive integers t.

Now we shall take free generators $x_1, \dots, x_{s-1}, x_{01}$, and x_{02} of the relatively free algebra $F(\mathrm{var}(L_s))$. Recall that the capital letter denotes an appropriate inner derivation of the algebra. Denote

$$R_k = [\dots[\bar{X}_1, \underbrace{X_{01}], \dots X_{01}]}_{k-2}[\dots[\bar{X}_2, \underbrace{X_{01}], \dots X_{01}]}_{k-3}\dots[\bar{X}_{k-2}, \bar{X}_{01}]\bar{X}_{k-1},$$

where $k = 3, \ldots, s$. Let also $R_1 = X_{01}$, $R_2 = X_1$. For example, $R_3 = [\bar{X}_1, \bar{X}_{01}]\bar{X}_2$ and $R_4 = [[\bar{X}_1, X_{01}], X_{01}][\bar{X}_2, \bar{X}_{01}]\bar{X}_3$.

Consider the following element of the relatively free algebra $F(\mathrm{var}(L_s))$, using for simplicity α_i instead of $\alpha_i^{(j)}$

$$g_t = x_{02} R_1^{(\alpha_1 - \sum_{i=2}^{s}(i-2)\cdot\alpha_i)nt} R_2^{(\alpha_2-\alpha_3)nt} \cdots R_k^{(\alpha_k-\alpha_{k+1})nt} \cdots R_s^{\alpha_s nt}.$$

In the element g_t we additionally alternate the variable X_1 from R_2 with X_{01} from R_k, where $4 \le k \le s$. Since $\sum_{i=1}^{s}(2-i)\cdot\alpha_i = 0$, we obtain that

$$g_t = x_{02} R_2^{(\alpha_2-\alpha_3)nt} \cdots R_k^{(\alpha_k-\alpha_{k+1})nt} \cdots R_s^{\alpha_s nt}.$$

Remark that if $s = 3$ then $\alpha_1 = \alpha_2 = \alpha_3 = \dfrac{1}{3}$ and $g_t = x_{02}\left([\bar{X}_1, \bar{X}_{01}]\bar{X}_2\right)^{\frac{nt}{3}}$ without the additional alternating pair X_1, X_{01}.

The degree of g_t is $m = nt + 1$, because the degree of R_k, $k \ge 2$, equals $k \cdot (k-1)/2$.

Let f_t be the complete linearization of the element g_t, and let V_t be the submodule of the S_{nt+1}-module $P_{nt+1}(L_s)$, generated by f_t. The element g_t contains $\alpha_s nt$ alternating sets of s variables $\{x_{01}, x_1, x_2, \ldots, x_{s-1}\}$ and $(\alpha_i - \alpha_{i+1})nt$ alternating sets of i variables $\{x_{01}, x_1, \ldots, x_{i-1}\}$ in each, where $i = 2, \ldots, s-1$. All other variables, except x_{02}, which are not included in alternating sets, are equal to the same x_{01}. Therefore, the decomposition of the module V_t into a direct sum of irreducible components has only modules indexed by Young diagrams with $nt+1$ cells which contain a subdiagram corresponding to the partition $\lambda^{(j)} \cdot t \vdash nt$, that is consisting of nt cells.

We shall prove that at least one of these irreducible submodules of the module of multilinear polynomials $P_{nt+1}(L_s)$ is not zero. Consider the elements $h_i = x_{02}R_k$, $k = 2, \ldots, s$ and make the following substitution in $h_2, \ldots, h_s$

$$x_{02} = z_1 Z_{s-1}^m,\ x_1 = z_{s-1},\ x_2 = z_{s-2},\ \ldots,\ x_{s-1} = z_1,\ x_{01} = d.$$

If two elements z_i, z_j in the summation participate in the same commutator bracket, then such a term is zero, because M is a metabelian ideal of L. Hence only one from the $k!$ terms is not equal to zero and the result of this substitution is be equal to $z_1 Z_{s-1}^{k-1+m}$.

Thus, if in the element g_t we make the following substitution of elements of L

$$x_{02} = z_1 Z^m,\ x_1 = z_{s-1},\ x_2 = z_{s-2}, \ldots, x_s = z_1,\ x_{01} = d,$$

then the result of the substitution is not zero.

In this way we have proved that

$$\liminf_{t\to\infty} \sqrt[nt+1]{c_{nt+1}(L_s)} \geq F(s),$$

and, therefore, the inequality $\underline{\exp}(L_s) \geq F(s)$ holds.

To complete the proof we shall show that $\overline{\exp}(L_s) \leq F(s)$. Consider the sequence of partitions $\lambda^{(n)} \vdash n$, $\lambda^{(n)} \in M_n$, i.e., $\lambda^{(n)}$ is a partition with nonzero multiplicity. In this case, in particular, $\lambda^{(n)}_{s+1} \leq 1$ and $\lambda^{(n)}_{s+2} = 0$.

Define a constant $k = (s-1)\cdot(s-2)$ and a partition $\mu^{(n+k)} \vdash n{+}k$ in s parts by the following equalities: $\mu^{(n+k)}_1 = \lambda^{(n)}_1{+}k$, $\mu^{(n+k)}_i = \lambda^{(n)}_i$ for $i = 2,3,\ldots,s$. According to Lemma 1, the partition $\mu^{(n+k)}$ belongs to T_{n+k}. Note that the Young diagram corresponding to the partition $\lambda^{(n)}$ is a subdiagram of the diagram corresponding to the partition $\nu^{(n+k+1)} = (\mu^{(n+k)}_1, \ldots, \mu^{(n+k)}_s, 1)$ of $n{+}k{+}1$. Therefore, the next relations follow from the representation theory of the symmetric groups

$$d_{\lambda^{(n)}} \leq d_{\nu^{(n+k+1)}} \leq (n+k+1)\cdot d_{\mu^{(n+k)}}, \quad n = 1,2,\ldots.$$

Sincet k does not depend on n, we derive from these inequalities that

$$\limsup_{n\to\infty} \sqrt[n]{d_{\lambda^{(n)}}} \leq \limsup_{n\to\infty} \sqrt[n]{d_{\mu^{(n+k)}}} \leq \limsup_{n\to\infty} \sqrt[n+k]{d_{\mu^{(n+k)}}} \leq F(s).$$

Therefore $\overline{\exp}(L_s) \leq F(s)$, and the proof of the theorem is completed. □

4. Investigation of the function $F(\alpha_1,\ldots,\alpha_s)$. Now we shall prove the results about the maximum of the function $F(\alpha_1,\ldots,\alpha_s)$ in stated in the previous section.

Lemma 2. *If $s > 3$, then for the maximum value of the function $F(\alpha_1,\ldots,\alpha_s)$ all inequalities in the third condition of* (4) *are strict*

$$\alpha_1 > \cdots > \alpha_s > 0.$$

P r o o f. Instead of the function $F(\alpha_1,\ldots,\alpha_s)$ we can consider its logarithm

$$\ln(F(\alpha_1,\ldots,\alpha_s)) = -\alpha_1\cdot\ln(\alpha_1) - \cdots - \alpha_s\cdot\ln(\alpha_s).$$

We shall show that if some of the inequalities $\alpha_1 \geq \cdots \geq \alpha_s \geq 0$ in the third condition of (4) are equalities, then some of the variables α_i can be changed with saving the other conditions of (4) in such a way that the number of the equalities will decrease, and the value of $\ln(F)$ as well of F will increase. Thus after a finite number of changes we can remove all of the equalities increasing the value of the function in each step. Really, we can consider the few possible cases.

Case 1 (leftmost and center). There is a fragment of the form:

$$\alpha_{l-1} > \alpha_l = \cdots = \alpha_k > \alpha_{k+1} \geq 0, \text{ where } 1 \leq l < k < s \text{ and } \alpha_0 := 1 \text{ if } l = 1.$$

Then we change three of the variables α in the following way:

$$(\widetilde{\alpha}_l, \widetilde{\alpha}_k, \widetilde{\alpha}_{k+1}) = (\alpha_l + \beta, \alpha_k - 2\beta, \alpha_{k+1} + \beta).$$

If β is small and positive, the variation also respects (4). We consider F as a function of the variable β: $F = F(\beta)$. For small positive β the function $F(\beta)$ increases, because the function $\ln(F)$ and its derivative have the form

$$\begin{aligned}
\ln(F(\beta)) &= C - (\alpha_l + \beta) \cdot \ln(\alpha_l + \beta) - (\alpha_k - 2\beta) \cdot \ln(\alpha_k - 2\beta) \\
&\quad -(\alpha_{k+1} + \beta) \cdot \ln(\alpha_{k+1} + \beta), \\
(\ln(F(\beta)))' &= -\ln(\alpha_l + \beta) - 1 + 2\ln(\alpha_k - 2\beta) + 2 - \ln(\alpha_{k+1} + \beta) - 1 \\
&= \ln \frac{(\alpha_k - 2\beta)^2}{(\alpha_l + \beta) \cdot (\alpha_{k+1} + \beta)} = \ln \frac{(\alpha_k - 2\beta)^2}{(\alpha_k + \beta) \cdot (\alpha_{k+1} + \beta)}.
\end{aligned}$$

Hence $(\ln(F(\beta)))'_{\beta=0+} = \ln(\alpha_k/\alpha_{k+1}) > 0$, and for small positive β the function $F(\beta)$ increases (this is true also for $\alpha_{k+1} = 0$, when this derivative tends to $+\infty$).

Case 2 (right extreme trivial). There is a fragment of the form

$$\alpha_{k-2} > \alpha_{k-1} > \alpha_k > \alpha_{k+1} = \cdots = \alpha_s = 0, \text{ where } k \geq 2 \text{ and } \alpha_0 := 1 \text{ if } k = 2.$$

Then the variation $(\widetilde{\alpha}_{k-1}, \widetilde{\alpha}_k, \widetilde{\alpha}_{k+1}) = (\alpha_{k-1} + \beta, \alpha_k - 2\beta, \alpha_{k+1} + \beta)$ for small positive β respects (4). It also increases both $F(\beta)$ and the number of strict inequalities in the line.

Case 3 (the rightmost positive). There is a fragment of the form

$$\alpha_{k-2} \geq \alpha_{k-1} > \alpha_k = \cdots = \alpha_s > 0, \text{ where } 2 \leq k < s \text{ and } \alpha_0 := 1 \text{ if } k = 2.$$

The changing of the variables $(\widetilde{\alpha}_{k-1}, \widetilde{\alpha}_k, \widetilde{\alpha}_s) = (\alpha_{k-1} - \beta, \alpha_k + 2\beta, \alpha_s - \beta)$ respecting (4). Then the value of the function $F(\beta)$ increases for small positive β as its partial derivative is as follows

$$(\ln(F(\beta)))' = \ln \frac{(\alpha_{k-1} - \beta) \cdot (\alpha_s - \beta)}{(\alpha_k + 2\beta)^2} = \ln \frac{(\alpha_{k-1} - \beta) \cdot (\alpha_k - \beta)}{(\alpha_k + 2\beta)^2};$$

$$(\ln(F(\beta)))'_{\beta=0+} \quad = \quad \ln(\alpha_{k-1}/\alpha_k) > 0.$$

There are still two cases to be considered

- $\alpha_1 > \alpha_2 = \dots = \alpha_s = 0$;
- $\alpha_1 = \alpha_2 = \dots = \alpha_s > 0$.

In the former case, if $s \geq 2$, the function does not achieve its maximum

$$F(\overrightarrow{\alpha}) = F(1, 0, \dots) = 1 < 2 = F(0.5, 0.5, 0, \dots).$$

In the latter case we have $\alpha_i = 1/s$, and the second inequality of (4) implies the restriction $s \leq 3$

$$\sum_{i=1}^{s} (2-i) \cdot \alpha_i = \sum_{i=1}^{s} (2-i) \cdot \frac{1}{s} = \frac{3-s}{2} \geq 0.$$

Lemma 2 is proved. □

Remark. If $s = 1, 2, 3$ the condition (4) reduces to the first equality and the non-negativity of the variables. In these cases the maximum $F_{\max}$ is achieved for equal variables α_i, and it is 1, 2, and 3, respectively.

Corollary 1. *The maximum of $F(\alpha_1, \dots, \alpha_s)$ on the domain satisfying (4) is attained only at a point satisfying $\alpha_s > 0$. Therefore these maximums strictly increase with s:*

$$\max_{(4)} F(\alpha_1, \dots, \alpha_{s-1}, \alpha_s) > \max_{(4),\ \alpha_s=0} F(\alpha_1, \dots, \alpha_{s-1}, \alpha_s) = \max_{(4)} F(\alpha_1, \dots, \alpha_{s-1}).$$

Lemma 3. *When $s > 3$, then the condition (4) can be replaced by more exact*

$$(6) \qquad \begin{cases} \alpha_1 + \alpha_2 + \ldots + \alpha_s & = & 1 \\ \sum_{i=1}^{s} (2-i) \cdot \alpha_i & = & 0 \\ \alpha_1 > \alpha_2 > \ldots > \alpha_s & > & 0 \end{cases}$$

P r o o f. First, we shall find the maximum of $F(\alpha_1, \dots, \alpha_s) = \alpha_1^{-\alpha_1} \cdot \ldots \cdot \alpha_s^{-\alpha_s}$ subject to the single condition $\alpha_1 + \ldots + \alpha_s = 1$, $\alpha_i > 0$. We form the Lagrangian

$$L(\alpha_1, \dots, \alpha_s; \lambda) = \ln(F(\alpha_1, \dots, \alpha_s)) + \lambda \cdot (\alpha_1 + \dots + \alpha_s - 1),$$

and write the equations for stationarity

$$\begin{cases} \dfrac{\partial L}{\partial \alpha_i} = -\ln(\alpha_i) - 1 + \lambda & = & 0, \ i = 1, \dots, s \\ \alpha_1 + \alpha_2 + \dots + \alpha_s & = & 1 \end{cases}$$

Hence all α_i are equal and we have the only point of stationarity $\overrightarrow{\alpha}=(1/s, \dots, 1/s)$. Study it for extremality

$$d^2L(\overrightarrow{\alpha};\lambda) = -s\sum_{i=1}^{s} d\alpha_i^2,$$

$$d^2L(\overrightarrow{\alpha};\lambda)|_{\mathcal{T}} = -s\cdot\left(\sum_{i=1}^{s-1} d\alpha_i^2 + (-\sum_{i=1}^{s-1} d\alpha_i)^2\right) = -2s\cdot\sum_{1\le i\le j\le s-1} d\alpha_i\cdot d\alpha_j,$$

where $\mathcal{T}\colon d\alpha_1 + \dots + d\alpha_s = 0$ is the equation for the tangent space.

The $(s-1)\times(s-1)$-matrix of the last quadratic form is negative definite

$$-s\cdot\begin{pmatrix} 2 & 1 & \cdots & 1 \\ 1 & 2 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 2 \end{pmatrix} \sim -s\cdot\begin{pmatrix} s & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} < 0.$$

So $(1/s, \dots, 1/s)$ is the unique local maximum of F and $F(1/s, \dots, 1/s) = s$. But we have

$$\lim_{\alpha\to 0+} \alpha^{-\alpha} = 1, \quad \lim_{\alpha\to+\infty} \alpha^{-\alpha} = 0, \quad \alpha^{-\alpha} \le e^{1/e},$$

then on the boundary of the domain by the continuity the corresponding factors become unit, and the maximal value of F decreases. If some of the α_i grows infinitely the function F converges to zero. Hence, the obtained stationary point is a strict global maximum.

The domain $U\colon \sum_{i=1}^{s}(i-2)\cdot\alpha_i < 0$ for $s > 3$ does not include the extremal point $(1/s, \dots, 1/s)$. Thus, the local maximum of F for the conditions (4) belongs to the boundary of the domain U, – otherwise it would be another local extremum of the more general problem with the single restriction which we consider. So it satisfies (6) and Lemma 3 is proved. $\square$

Lemma 4. *If $s > 3$, then $\max\limits_{(6)} F(\alpha_1, \dots, \alpha_s) = s\cdot q^{2-s}/(2-q)$, where q is a root of the polynomial*

$$P(x) = -x^{s-1} + x^{s-3} + 2x^{s-2} + \dots + (s-1)x + (s-2).$$

P r o o f. The Lagrangian function of the problem has the form (6)

$$L(\alpha_1,\dots,\alpha_s;\lambda,\mu)=\ln(F(\alpha_1,\dots,\alpha_s))+\lambda\cdot(\alpha_1+\dots+\alpha_s-1)+\mu\cdot\sum_{i=1}^{s}(i-2)\cdot\alpha_i.$$

The equations of stationarity are

$$\begin{cases}\dfrac{\partial L}{\partial\alpha_i}=-\ln(\alpha_i)-1+\lambda+(i-2)\cdot\mu=0,\ i=1,\dots,s\\ \alpha_1+\alpha_2+\dots+\alpha_s=1\\ \sum\limits_{i=1}^{s}(i-2)\cdot\alpha_i=0\\ \alpha_1>\alpha_2>\dots>\alpha_s>0.\end{cases}$$

Hence

$$\frac{\partial L}{\partial\alpha_{i+1}}-\frac{\partial L}{\partial\alpha_i}=\ln(\alpha_i)-\ln(\alpha_{i+1})+\mu=\ln\frac{\alpha_i}{\alpha_{i+1}}+\mu=0.$$

Let us define the constant q

$$\frac{\alpha_1}{\alpha_2}=\frac{\alpha_2}{\alpha_3}=\dots=\frac{\alpha_{s-1}}{\alpha_s}=:q,\ \text{ where } q=e^{-\mu}>1,\ \text{ and } \alpha_i=q^{s-i}\cdot\alpha_s.$$

The condition on q is determined by the substitution of α_i in the second equation of (6)

$$\sum_{i=1}^{s}(i-2)\cdot\alpha_i=\sum_{i=1}^{s}(i-2)\cdot q^{s-i}\cdot\alpha_s=0,\ \alpha_s>0:$$

$$\sum_{i=1}^{s}(i-2)\cdot q^{s-i}=-q^{s-1}+q^{s-3}+2\cdot q^{s-4}+\dots+(s-2)=0.$$

The variable α_s is defined by the substitution of α_i in the first equation of (6)

$$\sum\nolimits_{i=1}^{s}\alpha_i=\sum\nolimits_{i=1}^{s}q^{s-i}\cdot\alpha_s=1:\ \alpha_s=\left(\sum\nolimits_{i=0}^{s-1}q^i\right)^{-1}=\frac{q-1}{q^s-1}.$$

The equation for q can be simplified if we increase its degree

$$0\ =\ (q-1)\cdot\sum_{i=1}^{s}(i-2)\cdot q^{s-i}=-q^s+\sum_{i=1}^{s-1}q^{s-i}-s+2,$$

$$\begin{aligned} 0 &= (q-1)^2 \cdot \sum_{i=1}^{s} (i-2) \cdot q^{s-i} = (q-1) \cdot (-q^s + \sum_{i=1}^{s-1} q^s - s + 2) \\ &= -q^{s+1} + 2q^s - (s-1) \cdot q + s - 2 = (q^s - 1) \cdot (2-q) - s(q-1). \end{aligned}$$

Note that $q > 1$ and therefore α_s has another form

$$\alpha_s = \frac{q-1}{q^s - 1} = \frac{2-q}{s}, \text{ in particular, } q < 2.$$

Then we can calculate $\max\limits_{(6)} F(\alpha_1, \ldots, \alpha_s)$

$$\begin{aligned} &\prod_{i=1}^{s} \alpha_i^{-\alpha_i} = \prod_{i<s} (q^i \alpha_s)^{-q^i \alpha_s} = q^{-\alpha_s \cdot \sum_{i<s} iq^i} \cdot \alpha_s^{-\alpha_s \cdot \sum_{i<s} q^i} \\ = \ & q^{2-s} \cdot \alpha_s^{-1} = \frac{1}{q^{s-2} \cdot \alpha_s} = \frac{s}{q^{s-2} \cdot (2-q)}. \end{aligned}$$

Here we have used the following equalities:

$$\begin{aligned} &\alpha_s \cdot \sum\nolimits_{i=1}^{s-1} iq^i = \alpha_s \cdot \left((s-1)q^s - \sum\nolimits_{i=1}^{s-1} q^i \right) / (q-1) \\ = \ & \alpha_s \cdot \left(sq^s - \sum\nolimits_{i=1}^{s} q^i \right) / (q-1) = \alpha_s \cdot \frac{sq^s}{q-1} - q\alpha_s \cdot \frac{q^{s-1} + \cdots + q + 1}{q-1} \\ = \ & \alpha_s \cdot \left(\frac{s(q^s-1)}{q-1} + \frac{s}{q-1} \right) - \frac{q}{q-1} = s + \frac{s}{q-1} \cdot \frac{2-q}{s} - \frac{q}{q-1} = s - 2. \end{aligned}$$

Lemma 4 is proved. □

Notation. Let $q(s)$ be a root of the equation

$$P_s(x) = \sum_{i=1}^{s} (i-2) \cdot x^{s-i} = -x^{s-1} + x^{s-3} + 2x^{s-4} + \cdots + (s-3)x + (s-2) = 0.$$

Note that the another equations satisfied by $q(s)$ have the forms:

$$-x^s + \sum_{i=1}^{s-1} x^i - s + 2 = 0 \quad \text{and} \quad -x^{s+1} + 2x^s - (s-1)x + s - 2 = 0.$$

Lemma 5. *If $s \geq 3$, then*

(1) *the equation $P_s(x) = 0$ has a unique positive solution;*

(2) *The sequence* $q(s)$, $s = 3, 4, \dots$, *belongs to* $[1, 2)$ *and strictly increases;*

(3) $\lim\limits_{s\to+\infty} q(s) = 2$.

Proof. (1) Clearly, $x = 0$ is not a solution. We rewrite the equation in the form

$$(1/x)^2 + 2(1/x)^3 + \ldots + (s-3)\cdot(1/x)^{s-2} + (s-2)\cdot(1/x)^{s-1} = 1.$$

The left side is strictly increasing for positive $1/x$, from $0+$ to $+\infty$, so the equality is realized in a unique $x = q(s)$.

(2) It is easy to calculate the values of $P_s(x)$ at the endpoints of $[1, 2]$

$$\begin{aligned} P_s(1) &= \sum_{i=1}^{s}(i-2)\cdot 1^{s-i} = \frac{s\cdot(s-2-1)}{2} \geq 0, \\ P_s(2) &= \sum_{i=1}^{s}(i-2)\cdot 2^{s-i} = \frac{-2^{s+1} + 2\cdot 2^s - (s-1)\cdot 2 + s - 2}{(2-1)^2} = -s < 0, \end{aligned}$$

so $q(s) \in [1, 2)$.

As we have already shown $y = 1/q(s)$ is a root of the equation

$$y^2 + 2y^3 + \ldots + (s-3)\cdot y^{s-2} + (s-2)\cdot y^{s-1} = 1.$$

Similarly, $z = 1/q(s+1)$ is a root of the equation

$$z^2 + 2z^3 + \ldots + (s-3)\cdot z^{s-2} + (s-2)\cdot z^{s-1} + (s-1)\cdot z^s = 1.$$

If for some s the inequality $q(s+1)^{-1} \geq q(s)^{-1}$ holds, then the expression depending on z exceeds the expression depending on y, but both are equal 1. Hence $q(s+1)^{-1} < q(s)^{-1}$ and $q(s{+}1) > q(s)$ for all $s \geq 3$.

(3) The sequence $q(s)$ is monotone and bounded, hence $q(+\infty) = \lim\limits_{s\to+\infty} q(s)$ exists and belongs to $(1, 2]$. Then $1/q(+\infty) \in [0.5, 1)$ is a solution of the equation:

$$1 = x^2 + 2x^3 + \cdots + (s-2)\cdot x^{s-1} + \cdots = x^2/(1-x)^2.$$

Consequently, $1/q(+\infty) = 0.5$ and $q(+\infty) = 2$.

This limit can be calculated also ina different way. The number $q(4)$ is the positive root of the equation

$$-x^3 + x + 2 = x + (2 - x^3) = 0,$$

so $2 - q(4)^3 < 0$ and for $s \geq 4$ we have $q(s) > \sqrt[3]{2}$.

Now, from the equation $-q(s)^{s+1} + 2q(s)^s - (s-1) \cdot q(s) + s - 2 = 0$ we obtain and evaluate

$$0 < 2 - q(s) = \frac{(s-1) \cdot q(s) - s + 2}{q(s)^s} < \frac{(s-1) \cdot 2 - s + 2}{2^{s/3}} = \frac{s}{2^{s/3}} \to 0.$$

This gives us the value of the required limit. Lemma 5 is proved. $\square$

Let $F(s) = \max_{(6)} F(\alpha_1, \ldots, \alpha_s)$ and $\alpha_i(s)$, $i = 1, \ldots, s$, be the extreme values of the variables.

Lemma 6.

(1) $\lim\limits_{s \to +\infty} F(s) = 4$;

(2) $3 = F(3) < \cdots < F(s) < F(s+1) < \cdots < 4$;

(3) $\lim\limits_{s \to +\infty} \alpha_i(s) = 2^{-i}$.

P r o o f. According to Lemma 5, $\lim\limits_{s \to +\infty} q(s) = 2$, and by Lemma 4:

$$\alpha_s(s) = \frac{q(s) - 1}{q(s)^s - 1}, \quad \alpha_i(s) = q(s)^{s-i} \cdot \alpha_s(s), \quad F(s) = q(s)^{2-s} \cdot \alpha_s(s)^{-1}.$$

(1) Then

$$\lim_{s \to +\infty} F(s) = \lim_{s \to +\infty} \frac{q(s)^2}{q(s) - 1} \cdot \frac{q(s)^s - 1}{q(s)^s} = \lim_{q \to 2} \frac{q^2}{q - 1} = 4.$$

(2) By Corollary 1, the sequence $F(s)$ strictly increases. Therefore it holds:

$$\sup_{s \geq 0} F(s) = \lim_{s \to +\infty} F(s) = 4.$$

(3) Using part 1, we deduce

$$\lim_{s \to +\infty} \alpha_i(s) = \lim_{s \to +\infty} \frac{q(s) - 1}{q(s)^i} \cdot \frac{q(s)^s}{q(s)^s - 1} = \lim_{q \to 2} \frac{q - 1}{q^i} = 2^{-i}.$$

Lemma 6 is proved. $\square$

Now the proposition from the previous section follows from Lemmas 2–6.

5. On the algebraicity of exponents.

Remark. According to Lemma 4, $F(s)$ is algebraic. As we have already proved, $F(3) = 3$. Using linear algebra or the Gröbner elimination of variables we can show that $F(4) \approx 3.61\ldots$ and $F(5) \approx 3.83\ldots$ are roots of the polynomials

$$4x^3 - 11x^2 - 8x - 16 \quad \text{and} \quad 27x^4 - 94x^3 - 15x^2 - 50x - 125, \text{ respectively.}$$

For larger s the algebraic expression for $F(s)$ is less elementary. For example, $F(6) \approx 3.92\ldots$ is a root of

$$256x^5 - 1077x^4 + 360x^3 - 108x^2 - 432x - 1296.$$

An explicit equation for $F(s)$, for all $s > 3$, can be found using the idea of the Cayley-Hamilton theorem:

Lemma 7. *Let $q \in \overline{\Phi}$ be a root of the polynomial $P(x) \in \Phi[x]$ of degree d and let $r = R(q)$, where $R(x) \in \Phi[x]$. Then*

(1) *r is algebraic over Φ of degree not greater than d;*

(2) *r is a root of the characteristic polynomial of the linear operator of multiplication by $R(x)$ in the algebra $\Phi[x]/(P(x))$;*

(3) *if $P(x) = P_1(x) \cdot P_2(x)$ is a product of two nontrivial relatively prime polynomials over Φ, then the characteristic polynomial (from the previous item) is reducible.*

Proof. (1) The powers $R^i(x)$, $i = 0, \ldots, d$ are linearly dependent modulo $P(x)$. Therefore, there exists a nontrivial linear combination:

$$a_0 \cdot R^0(x) + \cdots + a_d \cdot R^d(x) = P(x) \cdot S(x), \text{ where } S(x) \in \Phi[x]$$

with $a_i \in \Phi$. Hence

$$a_0 \cdot r^0 + \cdots + a_d \cdot r^d = a_0 \cdot R^0(q) + \cdots + a_d \cdot R^d(q) = P(q) \cdot S(q) = 0,$$

and r is a root of the polynomial $a_0 \cdot x^0 + \cdots + a_d \cdot x^d$.

(2) Let φ be the linear operator of multiplication by $R(x)$ in the algebra $\Phi[x]/(P(x))$

$$\varphi(T(x)) = R(x) \cdot T(x) \pmod{P(x)}, \quad T(x) \in \Phi[x],$$

and let $\chi_\varphi(t) = \det(\varphi - \mathrm{Id} \cdot t)$ be its characteristic polynomial. By the Cayley-Hamilton theorem we have:

$$\chi_\varphi(\varphi) = 0: \quad \chi_\varphi(R(x)) = \chi_\varphi(R(x)) \cdot 1 = 0 \pmod{P(x)}.$$

So there is an $S(x) \in \Phi[x]$, such that $\chi_\varphi(R(x)) = P(x) \cdot S(x)$ and

$$\chi_\varphi(r) = \chi_\varphi(R(q)) = P(q) \cdot S(q) = 0.$$

(3) According to the Chinese Remainder Theorem,

$$\Phi[x]/(P(x)) \cong \Phi[x]/(P_1(x)) \oplus \Phi[x]/(P_2(x)).$$

Consequently, over the field Φ there is a decomposition:

$$\chi_\varphi(t) = \chi_{\varphi_1}(t) \cdot \chi_{\varphi_2}(t),$$

where φ_i is the operator of multiplication by $R(x)$ in the algebra $\Phi[x]/(P_i(x))$. Lemma 7 is proved. $\square$

Calculation. Now we shall find a polynomial with integer coefficients, which annihilates $F(s)$, $s \geq 4$. First we take the polynomials from Lemma 4

$$\begin{aligned} P(x) &= (x-1)^2 \cdot \sum_{i=1}^{s} (i-2) \cdot x^{s-i} = -x^{s+1} + 2x^s - (s-1) \cdot x + s - 2, \\ R(x) &= x^{s-2} \cdot (2-x). \end{aligned}$$

We write the matrix of the operator of multiplication by $R(x)$ in the algebra $\Phi[x]/(P(x))$ with respect to the basis $\{1, x, \ldots, x^s\}$. It has a regular diagonal form

$$A_{\varphi,s} = \begin{pmatrix} 0 & 0 & 2-s & 0 & 0 & \ldots & \ldots & \ldots & 0 \\ 0 & 0 & s-1 & 2-s & 0 & \ddots & \ldots & \ldots & 0 \\ 0 & 0 & 0 & s-1 & 2-s & \ddots & \ddots & \ldots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 & s-1 & 2-s & 0 & 0 \\ 0 & 0 & \ddots & \ddots & \ddots & 0 & s-1 & 2-s & 0 \\ 2 & 0 & 0 & \ddots & \ddots & \ddots & 0 & s-1 & 2-s \\ -1 & 2 & 0 & 0 & \ddots & \ddots & \ddots & 0 & s-1 \\ 0 & -1 & 0 & 0 & 0 & \ddots & \ddots & \ddots & 0 \end{pmatrix}.$$

This $(s+1)\times(s+1)$ matrix has a similar form for $s=4$

$$A_{\varphi,4} = \begin{pmatrix} 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 3 & -2 & 0 \\ 2 & 0 & 0 & 3 & -2 \\ -1 & 2 & 0 & 0 & 3 \\ 0 & -1 & 0 & 0 & 0 \end{pmatrix}.$$

The characteristic polynomial of φ has the form

$$\chi_{\varphi,s}(t) = \det \begin{pmatrix} A & 0 & C & \dots & 0 & 0 & 0 \\ 0 & A & B & \dots & 0 & 0 & 0 \\ 0 & 0 & A & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 2 & 0 & 0 & \dots & A & B & C \\ -1 & 2 & 0 & \dots & 0 & A & B \\ 0 & -1 & 0 & \dots & 0 & 0 & A \end{pmatrix}, \quad \text{where } \begin{cases} A = -t \\ B = s-1 \\ C = 2-s \end{cases}.$$

We compute it, expanding the determinant along the rows (or the columns) containing not more than 2 non-zero elements. When $s \geq 5$, we obtain

$$\begin{aligned} \chi_{\varphi,s}(t) &= (-1)^s A \cdot \Delta(s-1) + 2(-1)^s A^2 \cdot \Delta(s-2) + (-1)^{s-1} A^2 C \cdot \Delta(s-3) \\ &+ 2(-1)^{s-1} A^3 C \cdot \Delta(s-4) + A^{s+1} + C^{s-2}(4A+2B+C). \end{aligned}$$

Here $\Delta(n)$ denotes the determinant of the regular $n \times n$ matrix of the form

$$\Delta(n) = \det \begin{pmatrix} B & C & \dots & 0 & 0 \\ A & B & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & B & C \\ 0 & 0 & \dots & A & B \end{pmatrix}.$$

It is defined by the recurrence relation

$$\Delta(n) = B \cdot \Delta(n-1) - AC \cdot \Delta(n-2): \quad \Delta(0) = 1,\ \Delta(1) = B,\ \Delta(2) = B^2 - AC.$$

Hence we obtain an expression for the generating function of the sequence $(\Delta(n))$

$$(1 - B \cdot z + AC \cdot z^2) \cdot \sum_{n \geq 0} \Delta(n) \cdot z^n = \Delta(0) + (\Delta(1) - B \cdot \Delta(0)) \cdot z$$

$$+ \sum_{n\geq 2}(\Delta(n) - B\cdot\Delta(n-1) + AC\cdot\Delta(n-2))\cdot z^n = 1.$$

Consequently,

$$\begin{aligned}
\sum_{n\geq 0}\Delta(n)\cdot z^n &= \frac{1}{(1-B\cdot z+AC\cdot z^2)} = \frac{1}{(1-z\cdot(B-AC\cdot z))} \\
&= \sum_{k\geq 0} z^k\cdot(B-AC\cdot z)^k = \sum_{k\geq 0} z^k\cdot\sum_{m=0}^{k}\binom{k}{m}\cdot z^m\cdot(-AC)^m\cdot B^{k-m} \\
&= \sum_{k\geq 0}\sum_{m=0}^{k} z^{k+m}\cdot\binom{k}{m}\cdot(-AC)^m\cdot B^{k-m} \\
&= \sum_{n\geq 0} z^n\cdot\sum_{m=0}^{[n/2]}\binom{n-m}{m}\cdot z^m\cdot(-AC)^m\cdot B^{n-2m}.
\end{aligned}$$

Hence we obtain the following explicit expression for $\Delta(n)$

$$\Delta(n)=\sum_{m=0}^{[n/2]}\binom{n-m}{m}\cdot z^m\cdot(-AC)^m\cdot B^{n-2m}=\sum_{m=0}^{n}\binom{n-m}{m}\cdot z^m\cdot(-AC)^m\cdot B^{n-2m}.$$

Here we have assumed that

$$\binom{k}{m} = \frac{k\cdot(k-1)\cdot\ldots\cdot(k-m+1)}{m!} = 0, \text{ for integers } 0\leq k<m.$$

In particular,

$$\binom{s-m}{m} = 0 \text{ for the integer } s \text{ with } s\geq m>[s/2]\geq 0.$$

We now return to the expression for $\chi_{\varphi,s}(t)$. Using the above recursion, we find the formula which holds also for $s\geq 4$

$$\begin{aligned}
\chi_{\varphi,s}(t) &= A^{s+1}+C^{s-2}\cdot(4A+2B+C)+(-1)^s A\cdot(4A+B)\cdot\Delta(s-2) \\
&+ (-1)^{s-1}A^2\cdot\Delta(s-3).
\end{aligned}$$

Substituting the value of $\Delta(n)$ and the expressions for A, B, C, after some simplifications, we have an expression:

$$\chi_{\varphi,s}(t) = (-t)^{s+1}+(2-s)^{s-2}\cdot(s-4t)$$

$$- \quad (-1)^s t \cdot s \cdot \sum_{0 \le m \le [s/2]} t^m \cdot \binom{s-m}{m} \cdot \frac{(2-s)^m \cdot (s-1)^{s-2m}}{(s-m)(s-m-1)}.$$

The number $q(s)^{s-2} \cdot (2 - q(s))$ is a root of the polynomial $(-1)^{s+1} \cdot \chi_{\varphi,s}(t)$

$$(s-2)^{s-2} \cdot (4t - s) + t \cdot s \cdot \sum_{0 \le m \le [s/2]} t^m \cdot \binom{s-m}{m} \cdot \frac{(2-s)^m \cdot (s-1)^{s-2m}}{(s-m)(s-m-1)} + t^{s+1}.$$

According to Lemma 7, this polynomial has another double root corresponding $x = 1$

$$t = 1^{s-2} \cdot (2-1) = 1.$$

To get rid of the factor $(t-1)^2$ in $(-1)^{s+1} \cdot \chi_{\varphi,s}(t)$, we need preliminary combinatorial formulas.

Lemma 8. *The following formulas hold for $s \ge 4$, but some of them are true after reductions for smaller s:*

1) $\displaystyle \binom{s-m}{m} = \frac{4^m}{2^s} \cdot \sum_{u \ge 0} \binom{s+1}{2u+1} \cdot \binom{u}{m}$, *and actually we have $m \le u \le [s/2]$;*

2) $$\sum_{\sigma \ge m \ge 0} \binom{\sigma - m}{m} \cdot \left(\frac{2-s}{(s-1)^2} \right)^m = \frac{(s-2)^{\sigma+1} - 1}{(s-3) \cdot (s-1)^\sigma};$$

3) $$\sum_{s \ge m \ge 0} \binom{s-m}{m} \cdot (2-s)^m \cdot (s-1)^{s-2m} = \frac{(s-2)^{s+1} - 1}{s-3};$$

4) $$\sum_{s > m \ge 0} m \cdot \binom{s-m}{m} \cdot \frac{(2-s)^m \cdot (s-1)^{s-2m}}{s-m} = \frac{(s-2) \cdot (1 - (s-2)^{s-1})}{s-3};$$

5) $$\sum_{s > m \ge 0} \binom{s-m}{m} \cdot \frac{(2-s)^m \cdot (s-1)^{s-2m}}{s-m} = \frac{(s-2)^s + 1}{s};$$

6) $$\sum_{m=0}^{s-2} m \cdot (m-1) \cdot \binom{s-m}{m} \cdot \frac{(2-s)^m \cdot (s-1)^{s-2m}}{(s-m) \cdot (s-m-1)} = \frac{(s-2)^2 \cdot ((s-2)^{s-3} - 1)}{s-3};$$

7) $$\sum_{s-1>m\geq 0} m\cdot\binom{s-m}{m}\cdot\frac{(2-s)^m\cdot(s-1)^{s-2m}}{(s-m)\cdot(s-m-1)}=-(s-2)^{s-2}-1;$$

8) $$\sum_{s-1>m\geq 0} m\cdot\binom{s-m}{m}\cdot\frac{(2-s)^m\cdot(s-1)^{s-2m}}{s-m-1}=\frac{1-(s-2)^{s-2}\cdot(s^2-3s+1)}{s-3};$$

9) $$\sum_{s-1>m\geq 0} \binom{s-m}{m}\cdot\frac{(2-s)^m\cdot(s-1)^{s-2m}}{s-m-1}=(s-3)\cdot(s-2)^{s-2};$$

10) $$\sum_{s-1>m\geq 0} \binom{s-m}{m}\cdot\frac{(2-s)^m\cdot(s-1)^{s-2m}}{(s-m)\cdot(s-m-1)}=\frac{(s-4)\cdot(s-2)^{s-2}-1}{s}.$$

P r o o f. 1) The binomial coefficients are involved in the series in two variables

$$\sum_{s\geq 0} y^s\sum_{m=0}^{s}\binom{s-m}{m}x^m=\sum_{k,m\geq 0}\binom{k}{m}x^m y^{k+m}=\sum_{k\geq 0}y^k\sum_{m\geq 0}\binom{k}{m}x^m y^m$$

$$=\sum_{k\geq 0}y^k\cdot(1+xy)^k=\frac{1}{1-y-xy^2}=\frac{1}{2\sqrt{1+4x}}\cdot\left(\frac{1+\sqrt{1+4x}}{1-\frac{1+\sqrt{1+4x}}{2}y}-\frac{1-\sqrt{1+4x}}{1-\frac{1-\sqrt{1+4x}}{2}y}\right)$$

$$=\frac{1}{\sqrt{1+4x}}\cdot\sum_{s\geq 0}\left(\left(\frac{1+\sqrt{1+4x}}{2}\right)^{s+1}-\left(\frac{1-\sqrt{1+4x}}{2}\right)^{s+1}\right)y^s.$$

It follows

$$\sum_{m=0}^{s}\binom{s-m}{m}x^m=\frac{1}{\sqrt{1+4x}}\cdot\left(\left(\frac{1+\sqrt{1+4x}}{2}\right)^{s+1}-\left(\frac{1-\sqrt{1+4x}}{2}\right)^{s+1}\right)$$

$$=\frac{1}{2^{s+1}\cdot\sqrt{1+4x}}\cdot\sum_{k\geq 0}\binom{s+1}{k}\cdot(\sqrt{1+4x})^k\cdot(1^k-(-1)^k)$$

$$=\frac{1}{2^s\cdot\sqrt{1+4x}}\cdot\sum_{u\geq 0}\binom{s+1}{2u+1}\cdot(\sqrt{1+4x})^{2u+1}=\frac{1}{2^s}\cdot\sum_{u\geq 0}\binom{s+1}{2u+1}\cdot(1+4x)^u$$

$$=\frac{1}{2^s}\cdot\sum_{u\geq 0}\binom{s+1}{2u+1}\sum_{m\geq 0}\binom{u}{m}\cdot 4^m x^m=\frac{1}{2^s}\cdot\sum_{m\geq 0}4^m x^m\sum_{u\geq 0}\binom{s+1}{2u+1}\cdot\binom{u}{m}.$$

The comparison of the coefficients of x^m gives us the desired equality 1).

Note that these binomial coefficients are involved in the interesting series

$$\sum_{s\geq m\geq 0}\binom{s-m}{m}\cdot(-1)^m=\frac{2}{\sqrt{3}}\cdot\sin\frac{(s+1)\pi}{3},\qquad \sum_{s\geq m\geq 0}\binom{s-m}{m}=F_s,$$

where F_s are the Fibonacci numbers $1,1,2,3,\ldots$.

To obtain the equality 2) we use the previous identity

$$\sum_{m=0}^{\sigma}\binom{\sigma-m}{m}\cdot\left(\frac{2-s}{(s-1)^2}\right)^m=\sum_{m\geq 0}\frac{4^m}{2^\sigma}\cdot\sum_{u\geq 0}\binom{\sigma+1}{2u+1}\cdot\binom{u}{m}\cdot\left(\frac{2-s}{(s-1)^2}\right)^m$$

$$=\frac{1}{2^\sigma}\cdot\sum_{u\geq 0}\binom{\sigma+1}{2u+1}\cdot\sum_{m\geq 0}\binom{u}{m}\cdot\left(\frac{8-4s}{(s-1)^2}\right)^m=\frac{1}{2^\sigma}\cdot\sum_{u\geq 0}\binom{\sigma+1}{2u+1}\cdot\left(1+\frac{8-4s}{(s-1)^2}\right)^u$$

$$=\frac{1}{2^\sigma}\cdot\sum_{u\geq 0}\binom{\sigma+1}{2u+1}\cdot\left(\frac{s-3}{s-1}\right)^{2u}=\frac{1}{2^\sigma}\cdot\frac{s-1}{s-3}\cdot\sum_{u\geq 0}\binom{\sigma+1}{2u+1}\cdot\left(\frac{s-3}{s-1}\right)^{2u+1}$$

$$=\frac{1}{2^{\sigma+1}}\cdot\frac{s-1}{s-3}\cdot\left(\left(1+\frac{s-3}{s-1}\right)^{\sigma+1}-\left(1-\frac{s-3}{s-1}\right)^{\sigma+1}\right)=\frac{(s-2)^{\sigma+1}-1}{(s-3)\cdot(s-1)^\sigma}.$$

To prove 3) we use 2), assuming that $\sigma=s$

$$\sum_{m=0}^{s}\binom{s-m}{m}\cdot(2-s)^m\cdot(s-1)^{s-2m}=(s-1)^s\cdot\sum_{m=0}^{s}\binom{s-m}{m}\cdot\left(\frac{2-s}{(s-1)^2}\right)^m$$

$$=\frac{(s-2)^{s+1}-1}{s-3}=\sum_{k=0}^{s}(s-2)^k.$$

Similarly, to prove 4) we use 2) for $\sigma=s-2$

$$\sum_{s>m\geq 0}\frac{m}{s-m}\cdot\binom{s-m}{m}\cdot(2-s)^m\cdot(s-1)^{s-2m}$$

$$=\sum_{s>m\geq 1}\binom{s-m-1}{m-1}\cdot(2-s)^m\cdot(s-1)^{s-2m}$$

$$=(s-1)^{s-2}\cdot(2-s)\cdot\sum_{s-1>m\geq 0}\binom{s-2-m}{m}\cdot\left(\frac{2-s}{(s-1)^2}\right)^m$$

$$= \quad \frac{(s-2)\cdot(1-(s-2)^{s-1})}{s-3}.$$

To prove 5) we use the formulas 3) and 4), and the equality

$$\frac{1}{s-m} = \frac{1}{s}\cdot\left(1+\frac{m}{s-m}\right).$$

The identity 6) is similar to 4), with the additional substitution $\sigma = s-4$.
To prove 7) we use the formulas 4) and 6), if we observe that

$$\frac{m}{(s-m)\cdot(s-m-1)} = \frac{1}{s-2}\cdot\left(\frac{m}{s-m}+\frac{m\cdot(m-1)}{(s-m)\cdot(s-m-1)}\right).$$

To prove 8) we use the formula 6) and 7), applying the equality

$$\frac{m}{s-m-1} = (s-1)\cdot\frac{m}{(s-m)\cdot(s-m-1)} - \frac{m\cdot(m-1)}{(s-m)\cdot(s-m-1)}.$$

To prove 9) we use the formulas 3) and 8), and the equality

$$\frac{1}{s-m-1} = \frac{1}{s-1}\cdot\left(1+\frac{m}{s-m-1}\right).$$

The identity 10) follows from 5) and 9), due to the equality:

$$\frac{1}{(s-m)\cdot(s-m-1)} = \frac{1}{s-m-1} - \frac{1}{s-m}.$$

This completes the proof of the lemma. □

We use the above results to remove the factor $(t-1)^2$ from the polynomial $(-1)^{s+1}\cdot\chi_{\varphi,s}(t)$

$$P(t) = \frac{(-1)^{s+1}\cdot\chi_{\varphi,s}(t)}{(t-1)^2} = (-1)^{s+1}\cdot\chi_{\varphi,s}(t)\cdot\sum\nolimits_{k\geq 0} t^k\cdot(k+1)$$

$$= \quad -s\cdot(s-2)^{s-2}\cdot\sum\nolimits_{k\geq 0} t^k\cdot(k+1) + 4\cdot(s-2)^{s-2}\cdot\sum\nolimits_{k\geq 0} t^{k+1}\cdot(k+1)$$

$$+ \quad \sum_{k\geq 0} t^k\cdot(k+1)\cdot\sum\nolimits_{m\geq 0}^{[s/2]} t^{m+1}\cdot s\cdot\binom{s-m}{m}\cdot\frac{(2-s)^m\cdot(s-1)^{s-2m}}{(s-m)\cdot(s-m-1)}$$

$$+ \quad \sum\nolimits_{k\geq 0} t^{k+s+1}\cdot(k+1) = -\sum\nolimits_{k\geq 0} t^k\cdot(k+1)\cdot s\cdot(s-2)^{s-2}$$

$$
\begin{aligned}
&+ \sum\nolimits_{k\geq 1} 4\cdot t^k\cdot k\cdot (s-2)^{s-2} + \sum\nolimits_{k\geq s} t^k\cdot (k-s)\\
&+ \sum\nolimits_{k\geq 1}\sum\nolimits_{m=0}^{[s/2]} t^{k+m}\cdot k\cdot s\cdot \binom{s-m}{m}\cdot \frac{(2-s)^m\cdot (s-1)^{s-2m}}{(s-m)\cdot (s-m-1)}\\
&= -s\cdot (s-2)^{s-2} + \sum\nolimits_{k\geq 1} t^k\cdot (s-2)^{s-2}\cdot (4\cdot k - s\cdot (k+1)) + \sum\nolimits_{k\geq s} t^k\cdot (k-s)\\
&+ \sum_{k\geq 1} t^k\cdot s\cdot \sum_{m=0}^{\min([s/2],k)} (k-m)\cdot \binom{s-m}{m}\cdot \frac{(2-s)^m\cdot (s-1)^{s-2m}}{(s-m)\cdot (s-m-1)}\\
&= -s\cdot (s-2)^{s-2} + \sum\nolimits_{k\geq 1} t^k\cdot (s-2)^{s-2}\cdot (4\cdot k - s\cdot (k+1)) + \sum\nolimits_{k\geq s} t^k\cdot (k-s)\\
&+ \sum\nolimits_{k=1}^{[s/2]} t^k\cdot s\cdot \sum\nolimits_{m=0}^{k} (k-m)\cdot \binom{s-m}{m}\cdot \frac{(2-s)^m\cdot (s-1)^{s-2m}}{(s-m)\cdot (s-m-1)}\\
&+ \sum\nolimits_{k>[s/2]} t^k\cdot s\cdot \sum\nolimits_{m=0}^{[s/2]} (k-m)\cdot \binom{s-m}{m}\cdot \frac{(2-s)^m\cdot (s-1)^{s-2m}}{(s-m)\cdot (s-m-1)}\\
&= -s\cdot (s-2)^{s-2} + \sum\nolimits_{k=1}^{[s/2]} t^k\cdot \Bigg((s-2)^{s-2}\cdot (4\cdot k - s\cdot (k+1))\\
&+ s\cdot \sum\nolimits_{m=0}^{k} (k-m)\cdot \binom{s-m}{m}\cdot \frac{(2-s)^m\cdot (s-1)^{s-2m}}{(s-m)\cdot (s-m-1)}\Bigg)\\
&+ \sum\nolimits_{k>[s/2]}^{s-1} t^k\cdot \Bigg((s-2)^{s-2}\cdot (4\cdot k - s\cdot (k+1))\\
&+ s\cdot \sum\nolimits_{m=0}^{[s/2]} (k-m)\cdot \binom{s-m}{m}\cdot \frac{(2-s)^m\cdot (s-1)^{s-2m}}{(s-m)\cdot (s-m-1)}\Bigg)\\
&+ \sum\nolimits_{k\geq s} t^k\cdot \Bigg((s-2)^{s-2}\cdot (4\cdot k - s\cdot (k+1)) + k - s\\
&+ s\cdot \sum\nolimits_{m=0}^{[s/2]} (k-m)\cdot \binom{s-m}{m}\cdot \frac{(2-s)^m\cdot (s-1)^{s-2m}}{(s-m)\cdot (s-m-1)}\Bigg).
\end{aligned}
$$

Corollary 2. *The number $q(s)^{s-2}\cdot (2-q(s))$ is a root of the polynomial*

$$P(t) = -s\cdot (s-2)^{s-2} + \sum\nolimits_{k=1}^{[s/2]} t^k\cdot \Bigg((s-2)^{s-2}\cdot ((4-s)\cdot k - s)$$

$$+ \quad s\cdot\sum\nolimits_{m=0}^{k}(k-m)\cdot\binom{s-m}{m}\cdot\frac{(2-s)^m\cdot(s-1)^{s-2m}}{(s-m)\cdot(s-m-1)}\Bigg)$$
$$+ \quad \sum\nolimits_{k>[s/2]}^{s-1} t^k\cdot(s-k)\,.$$

And, finally, the exponent of $F(s) = s/(q(s)^{s-2}\cdot(2-q(s)))$ is a root of the polynomial $x^{s-1}\cdot P(s/x)/s$ obtained from the polynomial $P(t)$ given above by the substitution $x = s/t$:

$$(s-2)^{s-2}\cdot x^{s-1} - \sum\nolimits_{k=1}^{[s/2]} x^{s-1-k}\cdot s^{k-1}\cdot\Bigg((s-2)^{s-2}\cdot((4-s)\cdot k - s)$$
$$+ \quad s\cdot\sum\nolimits_{m=0}^{k}(k-m)\cdot\binom{s-m}{m}\cdot\frac{(2-s)^m\cdot(s-1)^{s-2m}}{(s-m)\cdot(s-m-1)}\Bigg)$$
$$- \quad \sum\nolimits_{k=[s/2]+1}^{s-1} x^{s-1-k}\cdot s^{k-1}\cdot(s-k)\,.$$

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