ISSN 0001-4346, Mathematical Notes, 2012, Vol. 92, No. 2, pp. 176–185. © Pleiades Publishing, Ltd., 2012.
Original Russian Text © A. B. Verevkin, 2012, published in Matematicheskie Zametki, 2012, Vol. 92, No. 2, pp. 192–201.

On a Matrix Nilpotent Filter

A. B. Verevkin*

Ul'yanovsk State University

Received December 9, 2010; in final form, May 19, 2011

Abstract—We define two families of homogeneous ideals of the algebra of polynomials generated by power entries of the general matrix and its operator invariants. We study the combinatorial characteristics of these ideals and, in greater detail, the case of second order.

DOI: 10.1134/S0001434612070206

Keywords: *nilpotent filter, commutative polynomial algebra, $\mathbb{Z}$-graded quotient algebra, Gröbner basis, Hilbert series, filtration, Hilbert polynomial.*

1. INTRODUCTION

Let $M = (x_{ij})$ be a square matrix of n^2 commuting variables $X = \{x_{ij} \mid i, j = 1, \dots, n\}$. Consider its powers $M^s = (f_{ij}^{(s)}(X))$, where $f_{ij}^{(s)}(X)$ are homogeneous polynomials in X of multiple degree s. In the algebra of commutative polynomials $A = k[X]$ over a field k, we define the homogeneous ideals

$$F^{(s)} = (f_{ij}^{(s)}(X) \mid i, j = 1, \dots, n), \qquad F^{(0)} = A,$$

and denote by $B^{(s)}$ the $\mathbb{Z}$-graded quotient algebra $A/F^{(s)}$.

Lemma 1. *For $t > s$, there exists the following embedding of ideals: $F^{(t)} \subsetneq F^{(s)}$.*

Proof. Note that the entries of the matrix M^s lie in $F^{(s)}$; therefore, $M^s = 0$ in $B^{(s)}$. If $t \geq s$, then $M^t = M^s \cdot M^{t-s} = 0 \cdot M^{t-s} = 0$ in $B^{(s)}$; therefore, the entries of M^t and the ideal $F^{(t)}$ generated by them lie in $F^{(s)}$. In the case of the strict inequality, $F^{(s)}$ contains sth power entries of not appearing in $F^{(t)}$. □

This lemma motivates the main definition of the paper, which follows.

Definition. The chain of ideals

$$\dots \subset F^{(s+1)} \subset F^{(s)} \subset \dots \subset F^{(1)} \subset F^{(0)} = A$$

is called a *matrix nilpotent filter* on the algebra $A = k[X]$.

Let us study the combinatorial properties of the entries of the filter and its successive quotients, including the algebras $B^{(s)}$. To this end, we shall use the Gröbner bases for ideals (see [1, 2.1–2.7, 3.3]; [2, Chap. 2]).

To describe the nilpotent filter, let us define the following auxiliary ideals of the algebra A:

$$G^{(s)} = (g^{(s-n+1)}(X), \dots, g^{(s-1)}(X), g^{(s)}(X)),$$

*E-mail: abverevkin@gmail.com

where $s \geq n-1$, and the $g^{(r)}(X)$ are homogeneous polynomials in X of multiple degree r which are the operator invariants of the matrix M and are defined for $r \geq 0$ by the expansion

$$\sum_{r \geq 0} g^{(r)}(X) \cdot \theta^r := (\det(E - M \cdot \theta))^{-1};$$

note that $g^{(0)}(X) = 1$ and $G^{(n-1)} = A$.

The polynomials $g^{(r)}(X)$ were studied in [3], where they were denoted by K_{r+n-1}. We were not able to find out whether ideals similar to $G^{(s)}$ have been studied earlier.

In the following section, we shall try to find the properties of the ideals constructed in the most general case. For example, it will be shown in Lemma 3 that, for large s, $G^{(s)}$ specifies, on the algebra A, a filtration interlaced with the nilpotent filter $(F^{(s)})$.

The last section is devoted small dimensions $n = 1, 2, \dots$, and most statements there are of technical nature. The description of the Hilbert series of subfactors of the filtration constructed in Theorem 9 and the calculation of the Hilbert polynomial of the algebra $B^{(s)}$ in Lemma 11 are the concluding combinatorial results in this paper.

Structural algebraic and algebro-geometric questions merit a study in a separate paper, just as the study of similar problems in the noncommutative case; here these questions are not considered.

2. GENERAL RESULTS

Let us prove some general properties of the ideals constructed above.

Lemma 2. *The following embeddings of the ideals of the algebra A exist*:

1) $F^{(t+s)} \subseteq F^{(t)} \cdot F^{(s)}$ *and, in particular,* $F^{(s)} \subseteq (F^{(1)})^s$;

2) $\det M \cdot F^{(s)} \subseteq F^{(s+1)}$.

Proof. We have $M^{t+s} = M^t \cdot M^s$; therefore, the entries of M^{t+s} are expressed via the pairwise products of the entries of M^t and M^s, which gives the first assertion.

The matrix M is the root of the characteristic polynomial $\chi_M(\theta) = \det(M - \theta \cdot E_n)$; let us write out the equality $\chi_M(M) \cdot M^s = 0$:

$$(-1)^n M^{n+s} + (-1)^{n-1} \operatorname{tr} M \cdot M^{n-1+s} + \dots + \det M \cdot M^s = 0.$$

The matrix entries of the first n summands on the left-hand side lie in $F^{(s+1)}$; hence the entries of $\det M \cdot M^{(s)}$ also belong to $F^{(s+1)}$, which proves the second assertion. □

Lemma 3. *For $s \geq n-1$, the following embeddings of ideals exist*:

1) $\det M \cdot G^{(s)} \subseteq G^{(s+n)}$;

2) $F^{(s)} \subseteq G^{(s)}$;

3) *if $t \geq s$, then* $G^{(t)} \subseteq G^{(s)}$.

Proof. 1) Take the coefficients of θ^{s+n} in the equality

$$\det(E - M \cdot \theta) \cdot \sum_{s \geq 0} g^{(s)}(X) \cdot \theta^s = 1.$$

For $s \geq n-1$, they are of the form defining the recurrence relation on $(g^{(s)}(X))$:

$$g^{(s+n)}(X) - \operatorname{tr} M \cdot g^{(s+n-1)}(X) + \dots + (-1)^n \det M \cdot g^{(s)}(X) = 0. \tag{2.1}$$

By definition, the first n summands on the left-hand side lie in $G^{(s+n)}$; hence the last summand $\det(M)\cdot g^{(s)}(X)$ also belongs to $G^{(s+n)}$.

2) Consider the equalities

$$\sum_{s\ge 0} M^s\cdot\theta^s=(E_n-M\cdot\theta)^{-1}=\frac{1}{\det(E-M\cdot\theta)}\cdot(\widetilde{E_n-M\cdot\theta})^\top$$
$$=\Bigl(\sum_{t\ge 0} g^{(t)}(X)\cdot\theta^t\Bigr)\cdot(\widetilde{E_n-M\cdot\theta})^\top. \tag{2.2}$$

Here $(\widetilde{E_n-M\cdot\theta})$ denotes the matrix adjoint to $(E_n-M\cdot\theta)$ and consisting of $(n-1)\times(n-1)$ minors that are polynomials in θ (just as in X) of degree at most $n-1$. Therefore, our assertion follows from the relations:

$$f_{ij}^{(s)}(X)=\biggl(\Bigl(\sum_{t\ge 0} g^{(t)}(X)\cdot\theta^t\Bigr)\cdot(\widetilde{E_n-M\cdot\theta})_{ji}\biggr)_{\theta^s}\in(g^{(s-(n-1))}(X),\dots,g^{(s)}(X))=G^{(s)}.$$

3) The recurrence relation (2.1) implies the inclusion $g^{(s+n)}(X)\in G^{(s+n-1)}$, while, by definition, the following embedding holds:

$$\{g^{(s+n-1)}(X),g^{(s+n-2)}(X),\dots,g^{(s+1)}(X)\}\subset G^{(s+n-1)}.$$

Therefore, for $s\ge 0$, we have $G^{(s+n)}\subset G^{(s+n-1)}$, and we obtain the second homogeneous filtration on A:

$$\dots\subset G^{(t+1)}\subset G^{(t)}\subset\dots\subset G^{(n)}\subset G^{(n-1)}=A,\qquad t>n$$

$\square$

3. SMALL DIMENSIONS

More concrete statements on the filtrations constructed above can be proved for the small dimensions $n=1,2$.

The case $n=1$ is obvious: $F^{(s)}=G^{(s)}=(x^s)$, $B^{(s)}=k[x]/(x^s)=\langle 1,x,\dots,x^{s-1}\rangle_k$, while the $\mathbb{Z}$-graded A-module $F^{(s)}/F^{(s+1)}$ is isomorphic to $\langle x^s\rangle_k$.

The case $n=2$, which is studied in the remaining part of the paper is more complex. Let us adopt the more convenient notation:

$$M=\begin{pmatrix} x & y\\ z & v\end{pmatrix},\qquad \sum_{s\ge 0} g^{(s)}(X)\cdot\theta^s=(1-(x+v)\cdot\theta+(x\cdot v-y\cdot z)\cdot\theta^2)^{-1}.$$

For $s\ge 1$, the ideals $G^{(s)}=(g^{(s-1)}(X),g^{(s)}(X))$ were defined in the algebra $A=k[x,y,z,v]$.

Lemma 4. *In the case under consideration, the following formulas are valid*:

1) $g^{(s)}(X)=\displaystyle\sum_{t=\lceil s/2\rceil}^{s} C_t^{s-t}\cdot(-\det M)^{s-t}(\operatorname{tr} M)^{2t-s}$;

2) $F^{(s)}=\bigl(g^{(s)}(X)-g^{(s-1)}(X)\cdot v,\ g^{(s-1)}(X)\cdot y,\ g^{(s-1)}(X)\cdot z,\ g^{(s)}(X)-g^{(s-1)}(X)\cdot x\bigr)$;

3) $M^{s+1}=g^{(s)}(X)\cdot M-g^{(s-1)}(X)\cdot\det M\cdot E_2$.

Proof. Writing out the definition of $(g^{(s)}(X),\ s \ge 0)$ for $n = 2$, we obtain

$$\begin{aligned}\sum_{s\ge 0} g^{(s)}(X)\cdot\theta^s &= (1-\theta(\operatorname{tr} M - \det M\cdot\theta))^{-1} = \sum_{t\ge 0}\theta^t(\operatorname{tr} M - \det M\cdot\theta)^t\\ &= \sum_{t\ge 0}\theta^t\sum_{u=0}^{t} C_t^u(-\det M)^u(\operatorname{tr} M)^{t-u}\theta^u = \sum_{t\ge 0}\sum_{u=0}^{t}\theta^{t+u}C_t^u(-\det M)^u(\operatorname{tr} M)^{t-u}\\ &= \sum_{s\ge 0}\theta^s\sum_{t=0}^{s} C_t^{s-t}(-\det M)^{s-t}(\operatorname{tr} M)^{2t-s}.\end{aligned}$$

Combining this relation with the equality $C_t^{s-t} = 0$ for $2t < s$, we obtain the first formula.

In connection with the case under consideration, let us return to relation (2.2):

$$\begin{aligned}\sum_{s\ge 0} M^s\cdot\theta^s &= \biggl(\sum_{t\ge 0} g^{(t)}(X)\cdot\theta^t\biggr)\cdot(\widetilde{E_2 - M\cdot\theta})^\top\\ &= \biggl(\sum_{t\ge 0} g^{(t)}(X)\cdot\theta^t\biggr)\cdot\begin{pmatrix}1-v\cdot\theta & y\cdot\theta\\ z\cdot\theta & 1-x\cdot\theta\end{pmatrix}.\end{aligned}$$

Equating the coefficients of θ^s for $s > 0$, we find that

$$\begin{aligned}M^s &= \begin{pmatrix}g^{(s)}(X) - g^{(s-1)}(X)\cdot v & g^{(s-1)}(X)\cdot y\\ g^{(s-1)}(X)\cdot z & g^{(s)}(X) - g^{(s-1)}(X)\cdot x\end{pmatrix}\\ &= g^{(s)}(X)\cdot E_2 - g^{(s-1)}(X)\cdot\widetilde{M}^\top.\end{aligned}$$

Now the second formula follows from the first equality, while the third formula is obtained by multiplying the previous equalities by M. □

Examples. Formula 1) from Lemma 4 implies the following equalities:

- $g^{(0)}(X) = 1$;
- $g^{(1)}(X) = \operatorname{tr} M = x + v$;
- $g^{(2)}(X) = -\det M + (\operatorname{tr} M)^2 = x^2 + xv + v^2 + yz$;
- $g^{(3)}(X) = -2\det M\cdot\operatorname{tr} M + (\operatorname{tr} M)^3 = (x+v)(x^2+v^2+2yz)$.

Corollary 5. *For $s > 0$, item 2) of Lemma 4 implies*

$$\begin{aligned}&1)\quad \{g^{(s-1)}(X)\cdot y,\ g^{(s-1)}(X)\cdot z,\ g^{(s-1)}(X)\cdot(x-v)\}\subset F^{(s)};\\ &2)\quad g^{(s-1)}(X)\cdot v \equiv g^{(s-1)}(X)\cdot x \equiv g^{(s)}(X) \pmod{F^{(s)}};\\ &3)\quad G^{(s)}/F^{(s)} = A\cdot g^{(s-1)}(X).\end{aligned}$$

Lemma 6. *For $s > 0$, $t \ge 0$, the following congruences hold*:

$$\begin{aligned}&1)\quad g^{(t+s-1)}(X) \equiv x^t\cdot g^{(s-1)}(X) \pmod{F^{(s)}};\\ &2)\quad g^{(2s-1)}(X) \equiv 0 \pmod{F^{(s)}};\\ &3)\quad x^s\cdot g^{(s-1)}(X) \equiv 0 \pmod{F^{(s)}}.\end{aligned}$$

Proof. Let $u \geq s > 0$; then

$$g^{(u)}(X) - x \cdot g^{(u-1)}(X) \in F^{(u)} \subset F^{(s)};$$

therefore,

$$g^{(u)}(X) \equiv x \cdot g^{(u-1)}(X) \pmod{F^{(s)}},$$

and the first congruence follows from the equality

$$g^{(t+s-1)}(X) - x^t \cdot g^{(s-1)}(X) = \sum_{i=0}^{t-1} x^i \cdot (g^{(t+s-i-1)}(X) - x \cdot g^{(t+s-i-2)}(X)) \in F^{(s)}.$$

The entries of the matrix M^s lie in $F^{(s)}$; therefore, $\operatorname{tr} M^s \in F^{(s)}$. Let us express the matrix $M^{2s} = (M^s)^2$ in terms of the polynomials $g^{(u)}(X)$:

$$\begin{pmatrix} (g^{(s)})^2 - 2g^{(s)}g^{(s-1)}v + (g^{(s-1)})^2(v^2 + yz) & \operatorname{tr} M^s \cdot g^{(s-1)}y \\ \operatorname{tr} M^s \cdot g^{(s-1)}z & t(g^{(s)})^2 - 2g^{(s)}g^{(s-1)}x + t(g^{(s-1)})^2(x^2 + yz) \end{pmatrix}.$$

Therefore, the following relation holds in A:

$$\operatorname{tr} M^s \cdot g^{(s-1)}(X) \cdot y = g^{(2s-1)}(X) \cdot y,$$

which implies the second congruence

$$g^{(2s-1)}(X) = \operatorname{tr} M^s \cdot g^{(s-1)}(X) \in F^{(s)}.$$

The third congruence is a consequence of the previous two congruences:

$$x^s \cdot g^{(s-1)}(X) \equiv g^{(2s-1)}(X) \equiv 0 \pmod{F^{(s)}}.$$

$\square$

Theorem 7. *The A-module $G^{(s)}/F^{(s)}$, regarded as a $\mathbb{Z}$-graded space, is isomorphic to*

$$\langle g^{(s-1)}(X),\ x \cdot g^{(s-1)}(X),\ \dots,\ x^{s-1} \cdot g^{(s-1)}(X)\rangle_k.$$

Proof. By items 1) and 2) of Lemma 6, for $P(x, y, z, v) \in k[x, y, z, v]$ we have

$$P(x, y, z, v) \cdot g^{(s-1)}(X) \equiv P(x, 0, 0, x) \cdot g^{(s-1)}(X) \pmod{F^{(s)}}.$$

Taking into account item 3) of Corollary 5 and item 3) of Lemma 6, we obtain

$$G^{(s)}/F^{(s)} = \{P(x) \cdot g^{(s-1)}(X),\ \deg P(x) < s\}.$$

Thus, to prove the theorem, it remains to show that $x^{s-1} \cdot g^{(s-1)}(X) \notin F^{(s)}$ or, from congruence 1) of Lemma 6, $g^{(2s-2)}(X) \notin F^{(s)}$.

Define the ideal

$$J = (z) \lhd A = k[x, y, z, v]\colon\ A/J \cong k[x, y, v].$$

If $a \in A$, then $\overline{a} = a + J$ from A/J can be regarded as an element $k[x, y, v]$ if $z = 0$ is substituted.

Let us prove that

$$g^{(2s-2)}(X) + J = \overline{g^{(2s-2)}}(X) \not\subset F^{(s)} + J = \overline{F^{(s)}},$$

which implies the required. We have

$$\overline{M} = \begin{pmatrix} x & y \\ 0 & v \end{pmatrix} = \begin{pmatrix} \dfrac{1}{x - v} & -y \\ 0 & x - v \end{pmatrix} \begin{pmatrix} x & 0 \\ 0 & v \end{pmatrix} \begin{pmatrix} x - v & y \\ 0 & \dfrac{1}{x - v} \end{pmatrix}.$$

Therefore,

$$\begin{pmatrix} x & y \\ 0 & v \end{pmatrix}^s = \begin{pmatrix} \dfrac{1}{x-v} & -y \\ 0 & x-v \end{pmatrix} \begin{pmatrix} x^s & 0 \\ 0 & v^s \end{pmatrix} \begin{pmatrix} x-v & y \\ 0 & \dfrac{1}{x-v} \end{pmatrix} = \begin{pmatrix} x^s & y\cdot\dfrac{x^s-v^s}{x-v} \\ 0 & v^s \end{pmatrix}.$$

These equalities determine $\overline{f_{ij}^{(s)}}(X)$:

$$\overline{f_{11}^{(s)}}(X) = x^s, \qquad \overline{f_{22}^{(s)}}(X) = v^s, \qquad \overline{f_{12}^{(s)}}(X) = y\cdot\frac{x^s-v^s}{x-v}.$$

Let us fix the order $y > x > v$ and let us prove that the collection

$$\{\overline{f_{11}^{(s)}}(X), \quad \overline{f_{22}^{(s)}}(X), \quad \overline{f_{12}^{(s)}}(X)\}$$

is a Gröbner basis (see [1, item 2, pp. 35–37], [2, Chap. 2, p. 105 (Russian transl.)]) for the ideal

$$\begin{aligned} \overline{F^{(s)}} \lhd k[x,y,v]\colon\ \overline{f_{12}^{(s)}}(X) &= y(x^{s-1}+vx^{s-2}+\dots+v^{s-2}x+v^{s-1}) \\ &= \underline{x^{s-1}y} + vx^{s-2}y + \dots + v^{s-2}xy + v^{s-1}y. \end{aligned}$$

The leading monomial $\overline{f_{12}^{(s)}}(X)$ is $x^{s-1}y$ and, for $s \ge 2$, has the unique link $x\cdot x^{s-1}y = x^s\cdot y$ with x^s, the leading (and unique) monomial $\overline{f_{11}^{(s)}}(X)$. Hence we obtain just one composition; let us carry out its reduction:

$$\begin{aligned} x\cdot\overline{f_{12}^{(s)}}(X) - \overline{f_{11}^{(s)}}(X)\cdot y &= xy\frac{x^s-v^s}{x-v} - x^s y = y\,\frac{x^{s+1}-v^s x - x^{s+1}+vx^s}{x-v} \\ &= vxy\,\frac{x^{s-1}-v^{s-1}}{x-v} = vxy\cdot(x^{s-2}+vx^{s-3}+\dots+v^{s-3}x+v^{s-2}) \\ &= \underline{\underline{v x^{s-1}y}} + v^2x^{s-2}y+\dots+v^{s-2}x^2y+v^{s-1}xy \xrightarrow{f_{12}} vxy\frac{x^{s-1}-v^{s-1}}{x-v} - v\cdot y\frac{x^s-v^s}{x-v} \\ &= \frac{vy}{x-v}(x^s - v^{s-1}x - x^s + v^s) = \frac{v^s y\cdot(v-x)}{x-v} = -\underline{v^s}\cdot y \xrightarrow{f_{22}} 0. \end{aligned}$$

Thus, the unique composition of the collection

$$\{\overline{f_{11}^{(s)}}(X), \quad \overline{f_{22}^{(s)}}(X), \quad \overline{f_{12}^{(s)}}(X)\}$$

reduces to zero, and this set is a reduced set closed under composition, i.e., it is a Gröbner basis for the ideal $\overline{F^{(s)}}$. Strictly speaking, this argument was adjusted to the case $s \ge 4$, but it is easy to verify that, for $s = 2$ or 3, the result obeys the same formula.

For $s = 1$, there are no links between the leading monomials, and hence there are no compositions as well; therefore, the set in question is also a Gröbner basis.

The elements $\overline{g^{(s)}}(X)$ are given by the relation

$$\sum_{s\ge 0}\overline{g^{(s)}}(X)\cdot\theta^s = \frac{1}{\det(E_2-\overline{M}\cdot\theta)} = \frac{1}{(1-x\theta)(1-v\theta)} = \frac{1}{x-v}\left(\frac{x}{1-x\theta} - \frac{v}{1-v\theta}\right).$$

Hence we obtain the expressions $\overline{g^{(s)}}(X) = (x^{s+1}-v^{s+1})/(x-v)$ and the reduction of $\overline{g^{(2s-2)}}(X)$ to the normal form with respect to the Gröbner basis:

$$\begin{aligned} \overline{g^{(2s-2)}}(X) &= \frac{x^{2s-1}-v^{2s-1}}{x-v} \\ &= \underline{x^{2s-2}} + v\underline{x^{2s-3}} + \dots + v^{s-1}x^{s-1} + \dots + \underline{v^{2s-3}}x + \underline{v^{2s-2}} \xrightarrow{f_{11,22}} v^{s-1}x^{s-1} \ne 0. \end{aligned}$$

This argument is justified for $s \geq 3$; for $s = 2$,

$$\overline{g^{(2s-2)}}(X) = x^2 + vx + v^2 \to vx \neq 0;$$

for $s = 1$,

$$\overline{g^{(2s-2)}}(X) = 1 \neq 0.$$

Thus, always $\overline{g^{(2s-2)}}(X) \notin \overline{F^{(s)}}$, and Theorem 7 is proved. □

Theorem 8. *For $s \geq 1$, the sequence of $\mathbb{Z}$-graded A-modules is exact*:

$$0 \to A[-2s+1] \xrightarrow{\theta} A[-s] \oplus A[-s+1] \xrightarrow{\varphi} A \to A/G^{(s)} \to 0,$$

where

$$\theta(a) = (a \cdot g^{(s-1)}(X), a \cdot g^{(s)}(X)), \qquad \varphi((a', a'')) = a' \cdot g^{(s)}(X) - a'' \cdot g^{(s-1)}(X).$$

Proof. Obviously, $\varphi \circ \theta = 0$, and we need only to prove that $\ker \varphi = \operatorname{Im} \theta$. Let $(a', a'') \in \ker \varphi$ satisfy

$$a' \cdot g^{(s)}(X) - a'' \cdot g^{(s-1)}(X) = 0.$$

Then $a' \cdot g^{(s)}(X)$ is divisible by $g^{(s-1)}(X)$. Turning to the recurrence relation (2.1) for $s \geq 2$, we obtain

$$g^{(s)}(X) - \operatorname{tr} M \cdot g^{(s-1)}(X) + \det M \cdot g^{(s-2)}(X) = 0.$$

We find that $a' \cdot \det M \cdot g^{(s-2)}(X)$ is divisible by $g^{(s-1)}(X)$. Turning to the same recurrence relation (2.1) for $s \geq 3$, we obtain

$$g^{(s-1)}(X) - \operatorname{tr} M \cdot g^{(s-2)}(X) + \det M \cdot g^{(s-3)}(X) = 0.$$

We find that $a' \cdot (\det M)^2 \cdot g^{(s-3)}(X)$ is divisible by $g^{(s-1)}(X)$. Similarly, by induction on s, we can show that

$$a' \cdot (\det M)^{s-1} \cdot g^{(0)}(X) = a' \cdot (\det M)^{s-1} \vdots\, g^{(s-1)}(X).$$

By item 1) of Lemma 4 for $s \geq 2$, we have the equality

$$\begin{aligned} g^{(s-1)}(X) &= \sum_{t=\lceil (s-1)/2 \rceil}^{s-1} C_t^{s-1-t} \cdot (-\det M)^{s-1-t} (\operatorname{tr} M)^{2t-s+1} \\ &= (\operatorname{tr} M)^{s-1} - \det M \sum_{t=\lceil (s-1)/2 \rceil}^{s-2} C_t^{s-1-t} \cdot (-\det M)^{s-2-t} (\operatorname{tr} M)^{2t-s+1}. \end{aligned}$$

The polynomial $\operatorname{tr} M$ is irreducible in $A = k[x, y, z, v]$, and $\det M \notin (\operatorname{tr} M)$; hence $\det M$ and $\operatorname{tr} M$ are coprime. Therefore, $\det M$ has no common factors with $(\operatorname{tr} M)^{s-1}$ and $g^{(s-1)}(X)$, while $g^{(s-1)}(X)$ has no common factors with $(\det M)^{s-1}$. This implies that a' is divisible by $g^{(s-1)}(X)$, i.e., we have $a' = a \cdot g^{(s-1)}(X)$.

Then

$$a'' \cdot g^{(s-1)}(X) = a' \cdot g^{(s)}(X) = a \cdot g^{(s-1)}(X) \cdot g^{(s)}(X)$$

and $a'' = a \cdot g^{(s)}(X)$. Thus, we obtain the inclusion proving Theorem 8:

$$(a', a'') = (a \cdot g^{(s-1)}(X), a \cdot g^{(s)}(X)) = \theta(a) \in \operatorname{Im} \theta.$$

□

Definition (see [1, item 3, pp. 47–50], [2, Chap.. 9, item 3, p. 574 (Russian transl.)])**.** If $V \cong \bigoplus_{m \geq 0} V_m$ is a graded space over a field k, then its Hilbert function h_V is specified by the values $h_V(m) = \dim_k V_m$, and the Hilbert series $\sum_{m \geq 0} \dim_k V_m \cdot t^m$ is denoted by $V(t)$.

If there exists a polynomial whose values coincide with $h_V(m)$ for sufficiently large m, then it is denoted by $P_V(m)$ and is called the *Hilbert polynomial* of the space V.

Theorem 9. *For $s \geq 1$, the following equalities hold*:

$$\begin{aligned}
&1)\quad A/G^{(s)}(t) = \frac{(1-t^s)(1-t^{s-1})}{(1-t)^4};\\
&2)\quad G^{(s)}/F^{(s)}(t) = \frac{t^{s-1}(1-t^s)}{(1-t)};\\
&3)\quad B^{(s)}(t) = \frac{(1-t^{s-1})(1-t^s)}{(1-t)^4} + \frac{t^{s-1}(1-t^s)}{1-t};\\
&4)\quad G^{(s)}/G^{(s+1)}(t) = \frac{(1-t^s)(t^{s-1}+t^s)}{(1-t)^3};\\
&5)\quad F^{(s)}/F^{(s+1)}(t) = \frac{(1-t^s)(t^{s-1}+t^s)}{(1-t)^3} + t^{2s} + t^{2s-1} - t^{s-1}.
\end{aligned}$$

Proof. 1) Since $A(t) = (1-t)^{-4}$, Theorem 8 implies the equalities

$$A/G^{(s)}(t) = A(t)\cdot(1-(t^s+t^{s-1})+t^{2s-1}) = \frac{(1-t^s)(1-t^{s-1})}{(1-t)^4}.$$

2) Theorem 7 implies

$$G^{(s)}/F^{(s)}(t) = t^{s-1}+t^s+\cdots+t^{2s-2} = \frac{t^{s-1}(1-t^s)}{(1-t)}.$$

3) From 1) and 2), we obtain the equalities

$$B^{(s)}(t) = A/F^{(s)}(t) = A/G^{(s)}(t) + G^{(s)}/F^{(s)}(t) = \frac{(1-t^s)(1-t^{s-1})}{(1-t)^4} + \frac{t^{s-1}(1-t^s)}{1-t}.$$

4) It follows from 1) that

$$\begin{aligned}
G^{(s)}/G^{(s+1)}(t) = A/G^{(s+1)}(t) - A/G^{(s)}(t) &= \frac{(1-t^{s+1})(1-t^s)}{(1-t)^4} - \frac{(1-t^s)(1-t^{s-1})}{(1-t)^4}\\
&= \frac{(1-t^s)(t^{s-1}-t^{s+1})}{(1-t)^4} = \frac{(1-t^s)(t^{s-1}+t^s)}{(1-t)^3}.
\end{aligned}$$

5) From 3) and 4), we obtain the final equalities of Theorem 9:

$$\begin{aligned}
F^{(s)}/F^{(s+1)}(t) &= B^{(s+1)}(t) - B^{(s)}(t)\\
&= G^{(s)}/G^{(s+1)}(t) + G^{(s+1)}/F^{(s+1)}(t) - G^{(s)}/F^{(s)}(t)\\
&= \frac{(1-t^s)(t^{s-1}+t^s)}{(1-t)^3} + \frac{t^s(1-t^{s+1})}{1-t} - \frac{t^{s-1}(1-t^s)}{1-t}\\
&= \frac{(1-t^s)(t^{s-1}+t^s)}{(1-t)^3} - t^{s-1} + t^{2s-1} + t^{2s}.
\end{aligned}$$

□

Remark 10. The following two (bi)graded spaces are associated with the filtrations $(G^{(s)}, s \geq 1)$ and $(F^{(s)}, s \geq 0)$ on A:

$$A_G := \bigoplus_{s \geq 1} G^{(s)}/G^{(s+1)}, \qquad A_F := \bigoplus_{s \geq 0} F^{(s)}/F^{(s+1)}.$$

It is easy to find that their Hilbert series are

$$A_G(t,z) := \sum_{s \geq 1} G^{(s)}/G^{(s+1)}(t) \cdot z^s = \frac{(1+t)z}{(1-t)^2(1-tz)(1-t^2z)},$$

$$A_F(t,z) := \sum_{s \geq 0} F^{(s)}/F^{(s+1)}(t) \cdot z^s = \frac{(1+t)z}{(1-t)^2(1-tz)(1-t^2z)} + \frac{1-z}{(1-tz)(1-t^2z)}.$$

Lemma 11. *The algebras $B^{(s)} = A/F^{(s)}$ and $A/G^{(s)}$ possess identical Hilbert polynomials*

$$P_{B(s)}(m) = s(s-1)m - \frac{s(s-1)(2s-5)}{2}.$$

Proof. By items 2) and 3) of Theorem 9, the Hilbert functions of these algebras coincide for $m \geq 2s-1$; therefore, their Hilbert polynomials are equal. Item 1) of Theorems 9, implies the equalities

$$\begin{aligned}
A/G^{(s)}(t) &= \frac{(1-t^s)(1-t^{s-1})}{(1-t)^4} = (1 - t^{s-1} - t^s + t^{2s-1}) \cdot \sum_{m \geq 0} C_{-4}^m (-t)^m \\
&= \sum_{m \geq 0} (-1)^m C_{-4}^m (t^m - t^{m+s-1} - t^{m+s} + t^{m+2s-1}) \\
&= (\text{a polynomial of degree } 2s-2) \\
&\quad + \sum_{m \geq 2s-1} t^m \cdot ((-1)^m C_{-4}^m - (-1)^{m-s+1} C_{-4}^{m-s+1} - (-1)^{m-s} C_{-4}^{m-s}) \\
&\quad + \sum_{m \geq 2s-1} t^m \cdot (-1)^{m-2s+1} C_{-4}^{m-2s+1},
\end{aligned}$$

here

$$C_{-4}^k = (-4)(-5)\cdots(-4-k+1)\frac{1}{k!} = (-1)^k C_{k+3}^3 = (-1)^k \frac{(k+3)(k+2)(k+1)}{6}.$$

Therefore, for $m \geq 2s-1$, we obtain the following equalities proving Lemma 11:

$$\begin{aligned}
h_{B(s)}(m) &= C_{m+3}^3 - C_{m-s+4}^3 - C_{m-s+3}^3 + C_{m-2s+4}^3 \\
&= \frac{(m+3)(m+2)(m+1)}{6} - \frac{(m-s+4)(m-s+3)(m-s+2)}{6} \\
&\quad - \frac{(m-s+3)(m-s+2)(m-s+1)}{6} \\
&\quad + \frac{(m-2s+4)(m-2s+3)(m-2s+2)}{6} \\
&= s(s-1)m - \frac{s(s-1)(2s-5)}{2}.
\end{aligned}$$

□

Corollary 12. *Similarly, it can be proved that the graded A-modules $G^{(s)}/G^{(s+1)}$ and $F^{(s)}/F^{(s+1)}$ have identical Hilbert polynomials*

$$P(m) = 2sm - s(3s-4).$$

ACKNOWLEDGMENTS

The author wishes to express gratitude to A. V. Kondrat'ev for his program of computations in graded associative algebras GRAAL (UlGU-1998), which was used to verify the preliminary conjectures stated in the present paper.

This work was supported by the Russian Foundation for Basic Research (grant no. 10-01-00209-a).

REFERENCES

1. V. A. Ufnarovskii, "Combinatorial and asymptotic methods in algebra," in *Itogi Nauki i Tekhniki, Akad. Nauk SSSR, Current Problems in Mathematics. Fundamental Directions*, Vol. 57: *Algebra*-6 (VINITI, Moscow, 1990), pp. 5–177 [in Russian].
2. D. Cox, J. Little, and D. O'Shea, *Ideals, Varieties, and Algorithms: An Introduction to Computational Algebraic Geometry and Commutative Algebra*, 2nd ed. (Springer-Verlag, New York, 1997; Mir, Moscow, 2000).
3. L. M. Shifner, "On the power of a matrix," Mat. Sb. **42** (3), 385–394 (1935).